Section 10.2. New Polynomial Invariants

Note. In this section we mention the Jones polynomial and give a recursion formula for the HOMFLY polynomial. We give an example of the computation of a HOMFLY polynomial.

Note. Vaughan F. R. Jones introduced a new knot polynomial in: A Polynomial Invariant for Knots via von Neumann Algebra, *Bulletin of the American Mathematical Society*, **12**, 103–111 (1985). A copy is available online at the AMS.org website (accessed 4/11/2021). This is now known as the *Jones polynomial.* It is also a knot invariant and so can be used to potentially distinguish between knots. Shortly after the appearance of Jones' paper, it was observed that the recursion techniques used to find the Jones polynomial and Conway polynomial can be generalized to a 2-variable polynomial that contains information beyond that of the Jones and Conway polynomials.

Note. The 2-variable polynomial was presented in: P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, and A. Ocneanu, A New Polynomial of Knots and Links, *Bulletin of the American Mathematical Society*, **12**(2): 239–246 (1985). This paper is also available online at the AMS.org website (accessed 4/11/2021). Based on a permutation of the first letters of the names of the authors, this has become knows as the "HOMFLY polynomial."

Definition. The *HOMFLY polynomial*, $P_L(\ell, m)$, is given by the recursion formula

$$\ell P_{L_+}(\ell, m) + \ell^{-1} P_{L_-}(\ell, m) = -m P_{L_S}(\ell, m),$$

and the condition $P_U(\ell, m) = 1$ for U the unknot.

Note. The notation L_+ for a right handed crossing, L_- for a left handed crossing, and L_S for a smoothed crossing is introduced in the introduction to this chapter. Recall from Note 10.0.A that the recursion formula for the Alexander polynomial is $A_{L_+}(t) - A_{L_-}(t) = (1-t)A_{L_S}(t)$. From Note 10.1.A, the recursion relationship for the Conway polynomial is $\nabla_{L_+}(z) - \nabla_{L_-} = -z\nabla_{L_S}(z)$; examples on the use of this recursion relationship are given in Section 10.1.

Example. Consider again the unlink given in Figure 10.2.

Figure 10.2

As discussed in Example 10.1.C, in Figure 10.2 we have unknots (left and center) with a right-handed crossing (left, L_+), a left-handed crossing (center, L_-), and the same orientation with a smoothing (right, L_S; we denote this "unlink of two components" as U_2). So by the recursion formula for the HOMFLY polynomial

$$\ell P_{L_+}(\ell, m) + \ell^{-1} P_{L_-}(\ell, m) = -m P_{L_S}(\ell, m),$$

and condition $P_U(\ell, m) = 1$ for U the unknot, we have $\ell(1) + \ell^{-1}(1) = -mPL_S(\ell, m)$. Therefore the unlink of two components, U_2, has HOMFLY polynomial $P(\ell, m) =$

$-(\ell + \ell^{-1})/m$. If we set $\mu = -(\ell + \ell^{-1})/m$ then we can similarly show that the HOMFLY polynomial of the unlink of n components is μ^{n-1}.

Note. The computation of a HOMFLY polynomial is the same as the computation of the Conway polynomial (given in Section 10.1). In the exercises of this section, the following HOMFLY polynomials are to be calculated:

$$\begin{aligned} P_{3_1}(\ell, m) &= (-2\ell^2 - \ell^4) + \ell^2 m^2, \\ P_{4_1}(\ell, m) &= (-\ell^{-2} - 1 - \ell^2) + m^2, \\ P_{6_2}(\ell, m) &= (2 + 2\ell^2 + \ell^4) + (-1 - 3\ell^2 - \ell^4)m^2 + \ell^2 m^4. \end{aligned}$$

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