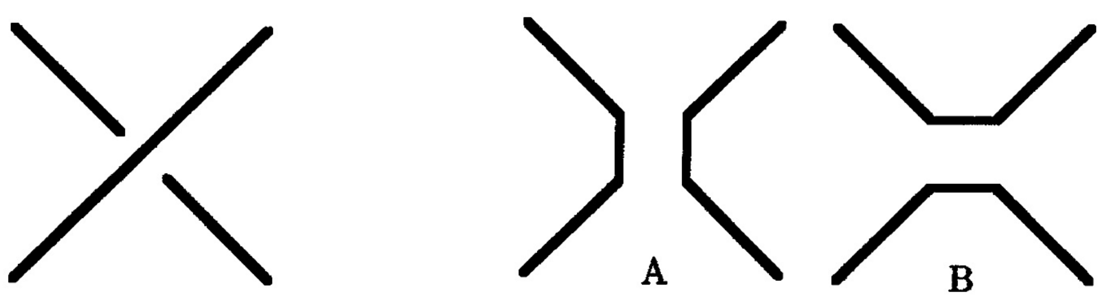
Section 10.3. Kauffman's Bracket Polynomial

Note. In this section we define two types of smoothings of crossings in a link diagram. Using these two choices of smoothings, we define a function $\langle D \rangle$. This function is not invariant under Reidemeister moves, so we define the Kauffman polynomial $F[K]$ in terms of $\langle D \rangle$. We discuss the Kauffman polynomial and give examples of its computation.

Note. The Kauffman polynmial and his bracket notation was introduced in: Louis H. Kaufmann, State Models and the Jones Polynomial, *Topology*, **26**(3), 395–407 (1987). Available online through Science Direct (accessed 4/11/2021).

Definition. In an (unoriented) link diagram D, crossings can be rotated to appear as in Figure 10.9(a). This crossing can then be smoothed in the two different ways given in Figure 10.9(b), called a *smoothing of type A* (Figure 10.9(b) left) and a *smoothing of type B* (Figure 10.9(b) right). A *state S* is a choice of smoothings for each of the crossings in the diagram.



(a) *Figure 10.9* (b)

Note. If a diagram has n crossings then there are 2^n possible states. For a given state with a crossings of type A and b crossings of type B, define $\langle D|S\rangle = t^{a-b}$. Now define

$$\langle D\rangle = \sum \langle D|S\rangle(-t^{-2} - t^2)^{|S|-1},$$

where the sum is taken over all 2^n states, and $|S|$ is the number of circles that result after all the smoothings of the given state are performed on the diagram (see the proof of Theorem 4.3.7 for the idea of a "circle").

Example. Consider the knot of Figure 10.10. Since it has 3 crossings, then it has $2^3 = 8$ states.

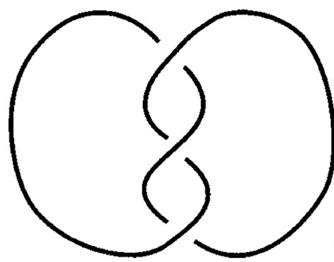
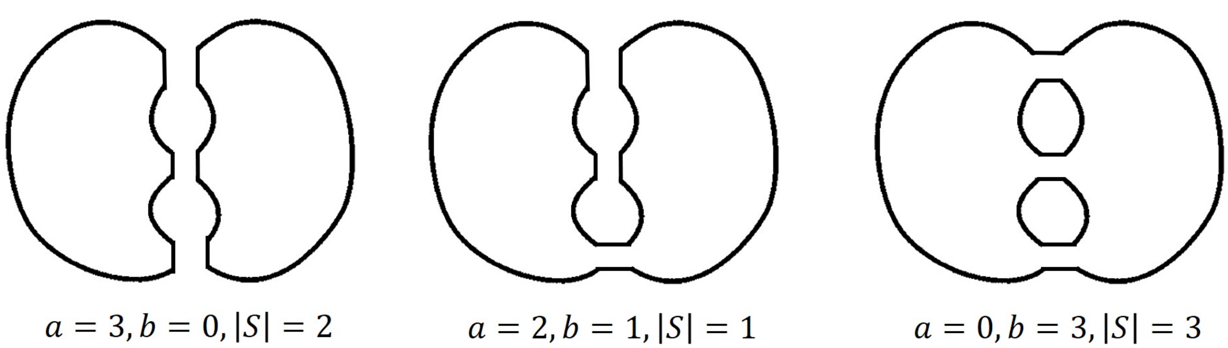


Figure 10.10

In the 8 states, we have 1 for which $a = 3$ and $b = 0$, 3 for which $a = 2$ and $b = 1$, 3 for which $a = 1$ and $b = 2$, and 1 for which $a = 0$ and $b = 3$. The number $|S|$ of circles in the resulting diagrams is 2, 1, 2, and 3, respectively (see below).



We then have

$$\langle D\rangle = \sum \langle D|S\rangle(-t^{-2}-t^{2})^{|S|-1} = \sum t^{a-b}(-t^{-2}-t^{2})^{|S|-1}$$

$$= (1)t^{(3)-(0)}(-t^{-2}-t^{2})^{(2)-1} + (3)t^{(2)-(1)}(-t^{-2}-t^{2})^{(1)-1} + (3)t^{(1)-(2)}(-t^{-2}-t^{2})^{(2)-1}$$

$$+(1)t^{(0)-(3)}(-t^{-2}-t^{2})^{(3)-1} = t^{3}(-t^{-2}-t^{2}) + 3t + 3t^{-1}(-t^{-2}-t^{2}) + t^{-3}(-t^{-2}-t^{2})^{2}$$

$$= -t - t^{5} + 3t - 3t^{-3} - 3t + t^{-3}(t^{-4}+2+t^{4}) = -t - t^{5} + 3t - 3t^{-3} - 3t + t^{-7} + 2t^{-3} + t$$

$$= -t^{5} - t^{-3} + t^{-7}.$$

(Notice that this differs from Livingston's result by two negative signs.)

Note. Recall the three Reidemeister moves from Figure 3.1:

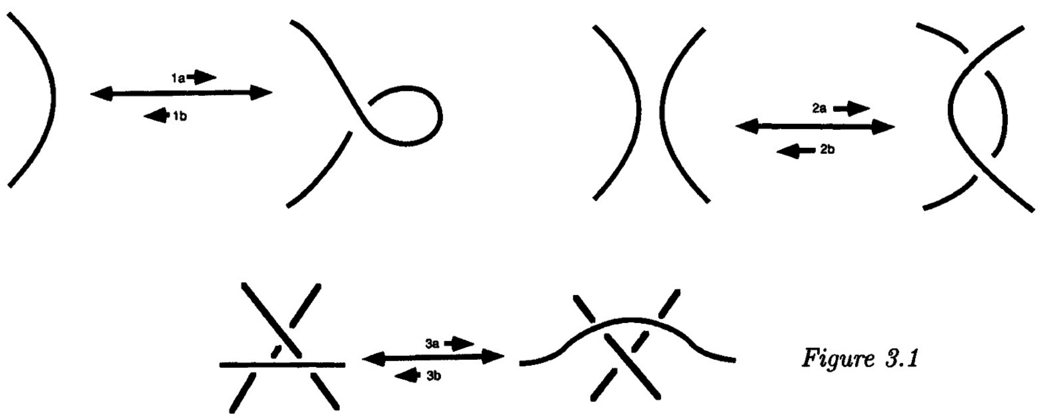


Figure 3.1

Livingston claims that the polynomial $\langle D\rangle$ can be shown to be invariant under Reidemeister moves 2 and 3. However, it can change under Reidemeister move 1, as can be seen by applying it to a diagram of the unknot. So polynomial $\langle D\rangle$ is not a knot invariant. We can use $\langle D\rangle$ to define a new polynomial which is a knot invariant (in particular, it is unchanged by Reidemeister move 1).

Definition. For a given link diagram D, orient each component. Let w denote the number of right-handed crossings minus the number of left-handed crossings (see Section 3.5. The Alexander Polynomial, Note 3.5.A for the definition of the handedness of crossings). The *Kauffman polynomial* $F[K]$ is $(-t)^{-3w}\langle D\rangle$.

Note. For the knot of Figure 10.10 above, we saw that $\langle D\rangle = -t^{-5} - t^{-3} + t^{-7}$ and we have $w = 3$, so the Kaufman polynomial is

$$(-t)^{-3w}\langle D\rangle = (-t)^{-9}(-t^5 - t^{-3} + t^{-7}) = t^{-4} + t^{-12} - t^{-16}.$$

(Notice that this agrees with Livingston.) Livingston claims that the exponents of the Kauffman polynomial are divisible by 4. Kauffman proved that the polynomial (in t and t^{-1}) $F[K](t^{-1/4})$ is in the Jones polynomial. Notice that for the knot of Figure 10.10, $F[K](t^{-1/4}) = (t^{-4} + t^{-12} - t^{-16})|_{t^{-1/4}} = (t^{-1/4})^{-4} + (t^{-1/4})^{-12} - (t^{-1/4})^{-16} = t + t^3 - t^4$, which is in fact the Jones polynomial of the trefoil (the knot of Figure 10.10 is actually just the trefoil knot 3_1).

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