Chapter 2. What is a Knot?

Note. In this chapter, we will give three definitions of a knot. We want to get to a rigorous formulation of knots, but still mostly concentrate on the "flavor of the problems" and so we retain a degree of informality.

Section 2.1. Wild Knots and Unknottings

Note. In this section we explore some attempted definitions of a knot and reveal the weaknesses of these attempts. In the next section, we give a formal definition.

Note. We might try to define a knot as the image of the closed interval $[0, 1]$ under a continuous function $f : \mathbb{R} \to \mathbb{R}^3$ which is one to one on $(0, 1)$ and $f(0) = f(1)$. That is, f takes the interval and embeds it in 3-space in such a way that the ends are connected and there are no other intersections of the image with itself. See Figure 2.1 which illustrates such a mapping.

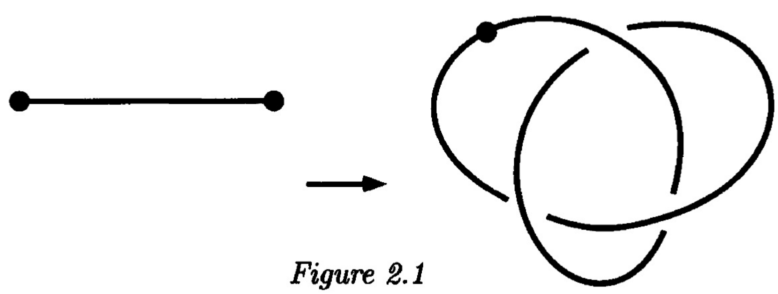

Figure 2.1

Note. However, the idea of simply continuously mapping $[0, 1]$ into $\mathbb{R}^3$ leads to possible undesirable behavior. In Figure 2.2 we have an infinitely knotted loop (this is referred to as a "wild knot"; a knot with a finite number of crossings is a "tame knot"). This is not what we intuitively think of as a knot and we will avoid such cases.

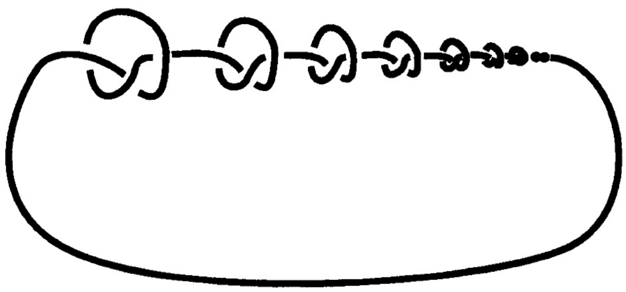

Figure 2.2

Note. Now, we turn our attention to the idea of deforming one knot into another knot. We might attempt to simply use continuous transformations of the knot, but Figure 2.3 illustrates a problem with this. In this figure, we continuous transform a trefoil knot to the trivial knot; this is undesirable and must be avoided!

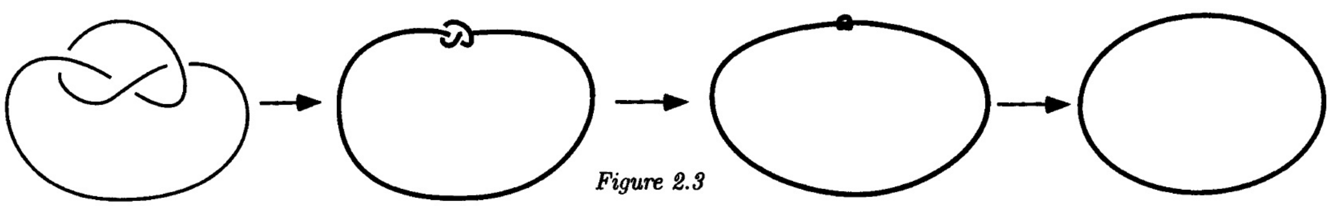

Figure 2.3

Note. One approach is to require that the mapping $f : [0,1] \to \mathbb{R}^3$ is differentiable (which is addressed in terms of a unit tangent vector to the knot). In this case the infinite knotted ("wild") knot of Figure 2.2 is not allowed (since the unit tangent vector oscillates without bound near the infinite knotting). We can also introduce differentiability into the setting of deformations but it is "more difficult." as Livingston says (see page 13).

Note. An alternative approach is to define a knot in terms of polygonal curves (that is, a finite collection of line segments joined end-to-end which determine a simple closed curve). We'll see this in the next section. These ideas are touched on in Appendix I, "Differentiable Knots are Tame," in: Richard Crowell and Ralph Fox, *Introduction to Knot Theory*, NY: Springer Verlag (1963); this source was mentioned in the Chapter 1 notes.

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