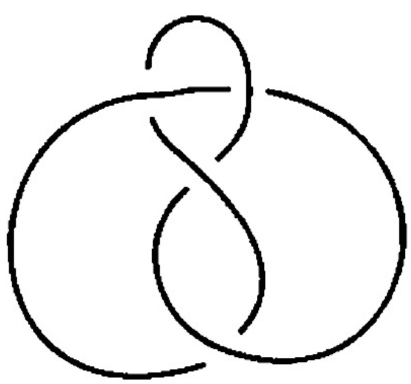
Section 2.4. Diagrams and Projections

Note. In this section we formally define a diagram and a projection of a knot. We state two theorems concerning small changes in the locations of vertices of a polygonal curve representing a knot. We state and prove our first theorem concerning the equivalence of knots (see Theorem 2.4.3).

Definition. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = (x, y)$ is the *projection map.* If K is a knot, then the image of K under this projection is called the *projection of knot K.*

Note. Below is the knot 4_1 from Appendix 1 (the "figure-8 knot"), along with a projection of it. Notice that we have lost the information about the overcrossings and undercrossings in the projection. So it is possible that different knots can have the same projection.



$\mathbf{4_1}$

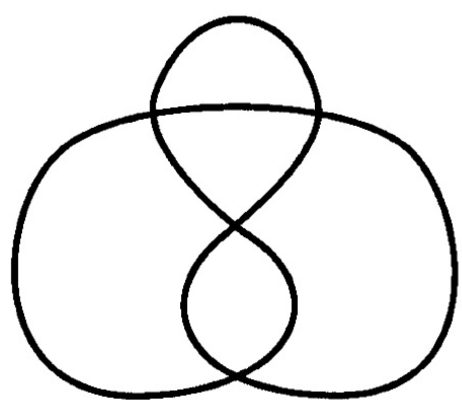


Figure 2.6

Definition. A *diagram* of a knot is the drawing of the map in two dimensions where overcrossings are indicated by solid segments and undercrossings are indicated by broken segments (as we have seen previously).

Note. As Livingston comments on page 21: "If two knots have the same diagram they are equivalent." So we make no distinction between a graph and its equivalence class, as commented earlier.

Note. We want a graph projection to preserve as much information as possible. In particular, we want to be able to distinguish vertices and segments. This leads to the following definition.

Definition. A knot projection is a *regular projection* if no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot.

Note. Not every knot (represented as a particular polygonal curve) has a regular projection. However, there is always an equivalent knot "nearby" that does have a regular projection. The nearby knot could result from very small displacements of the vertices of the particular knot. This is made quantitative in the next theorem, a proof of which is to be given in Exercise 2.4.2.

Theorem 2.4.1. Let K be a knot determined by the n-tuple of points $(p_1, p_2, \ldots, p_n)$. For every number $t > 0$ there is a knot K' determined by an ordered set $(q_1, q_2, \ldots, q_n)$ such that the distance from q_i to p_i is less than t for all $1 \leq i \leq n$, K' is equivalent to K, and the projection of K' is regular.

Note. The next theorem shows that if a knot has a regular projection, then all sufficiently small displacements of the vertices result in an equivalent knot with a regular projection. A proof of the theorem is to be given in Exercise 2.4.3.

Theorem 2.4.2. Suppose that K is determined by the sequence $(p_1, p_2, \ldots, p_n)$ and has a regular projection. There is a number $t > 0$ such that if a knot K' is determined by $(q_1, q_2, \ldots, q_n)$ with each q_i within a distance of t of p_i, then K' is equivalent to K and has a regular projection.

Note. The next result gives a general condition concerning the equivalence of knots.

Theorem 2.4.3. If knots K and J have regular projections and identical diagrams, then they are equivalent.

Note. Livingston gives a partial argument for Theorem 2.4.3 and leaves the details to Exercise 2.4.1.

Definition. In a knot diagram, the collection of arcs making up the diagram in the plane are called *edges* or *arcs* of the diagram. The points in the diagram which correspond to double points in the project are *crossing points* or *crossings*.

Note. A knot is a subset of 3-space as defined above. However, we will use terminology that does not distinguish between a knot, an equivalence class of knots, and a diagram of a knot (thanks to Theorem 2.4.3).

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