Section 2.5. Orientations

Note. In this brief section, we associate a "direction" with a knot determined by a polygonal curve with vertices in the n-tuple $(p_1, p_2, \ldots, p_n)$. Notice that if we apply a cyclic permutation to these vertices, we get the same knot; for example, $(p_2, p_3, \ldots, p_n, p_1)$ is the same polygonal curve (and so the same knot). In fact, we can reverse the order of the n-tuple to get another representation of the same polygonal curve (...and so the same knot): $(p_n, p_{n-1}, \ldots, p_2, p_1)$.

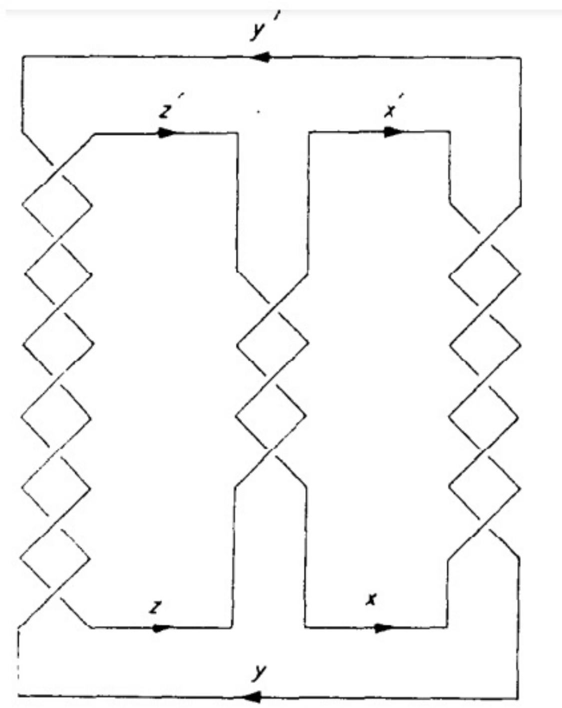
Definition. An *oriented knot* consists of a knot and an ordering of its vertices. The ordering must be chosen so that it determined the original knot. Two orderings are considered equivalent if they differ by a cyclic permutation.

Note. So we have that the orderings of the vertices of a polygonal curve $(p_1, p_2, \ldots, p_n)$ is different from the ordering $(p_n, p_{n-1}, \ldots, p_1)$. These two orderings represent the same polygonal curve (and so the same knot), but they represent potentially different oriented knots. When create a diagram for an oriented knot, we insert arrows on the segments to indicate the orientation.

Note. If a knot is oriented then an elementary deformation produces an oriented knot. So we can define equivalence of oriented knots the same way defined equivalent knots (namely, using a sequence of elementary deformations that deform one of the oriented knots into the other).

Definition. Oriented knots are *oriented equivalent* if there is a sequence of elementary deformations carrying one oriented knot to the other.

Note. Two oriented knots can be non-equivalent, yet still be equivalent (as knots, with the orientations ignored). The first such example was given by H. F. Trotter. He showed that the $(3,5,7)$-pretzel knot can be oriented in two was, by the resulting oriented knots are not oriented equivalent. Here is his Figure 1(a) (notice that the $(3,5,7)$-pretzel is equivalent to the $(5,7,3)$-pretzel and, in turn, equivalent to the $(7,3,5)$-pretzel by Exercise 1.2).



$p = 7,\ q = 3,\ r = 5$

These results appear in: H. F. Trotter, Non-invertible Knots Exist, *Topology*, **2**(4), 341–358 (1963); a copy can be viewed online at Science Direct (accessed 1/30/2021).

Note. We will make use of the following definitions later.

Definition. The *reverse* of the oriented knot determined by the n-tuple of vertices $(p_1, p_2, \ldots, p_n)$, is the oriented knot K^r determined by the n-tuple consisting of the same vertices but with their order reversed: $(p_n, p_{n-1}, \ldots, p_1)$. An oriented knot K is *reversible* if K and K^r are oriented equivalent. If K is not oriented, it is *reversible* if for some choice of orientation it is reversible.

Note. H. F. Trotter's general result in the paper mentioned above states (in terms of our definitions):

> **Theorem 1. (Trotter, 1963).** Let p, q, r be odd integers such that $|p|$, $|q|$, and $|r|$ are distinct and greater than 1. Then the (p, q, r)-pretzel in not reversible.

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