Chapter 3. Combinatorial Techniques

Note. "Combinatorial Techniques" for the study of knots are those techniques based on the properties of knot diagrams (such as the Alexander Polynomial, to be addressed in Section 3.5). In Chapter 10 we will consider "recent progress" in combinatorial knot theory (as Livingston says in his 1993 book), including the Conway polynomial and the Kaufman bracket polynomial.

Section 3.1. Reidemeister Moves

Note. In this section we give six manipulations of a knot diagram which we will show allow us to create different diagrams for the same knot (here, we think of a knot as an equivalence class of knots). We present that material by giving knot diagrams which are smooth (as opposed to projections of polygonal curves).

Note. We consider the following manipulations of a knot diagram. Notice that these manipulations are intuitively appealing in terms of what they represent as a manipulation of a knot thought of as a loop of string.

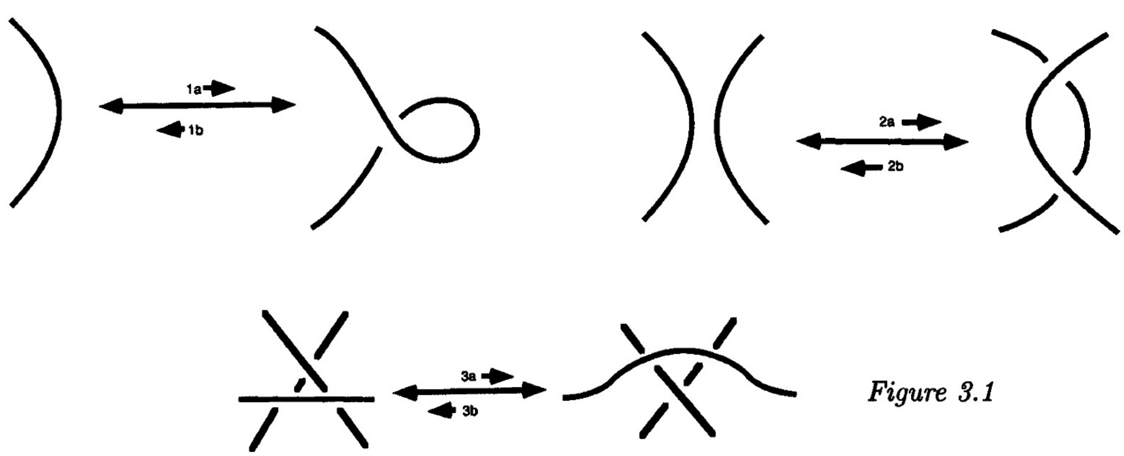
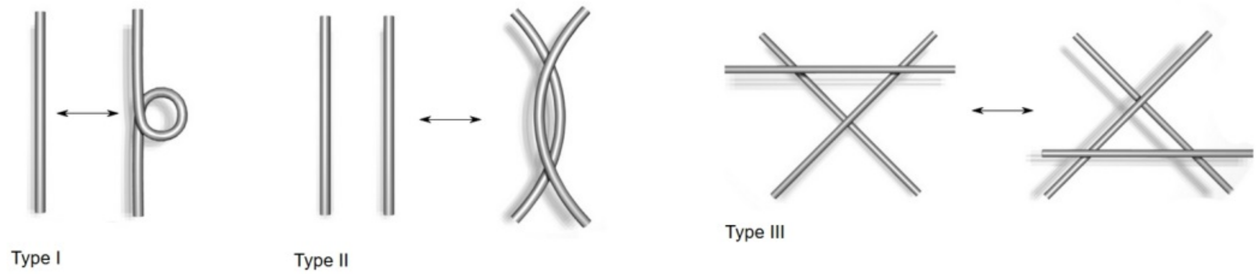


Figure 3.1

Note. A more three dimensional illustration of the Reidemeister moves can be found on WordPress.com (accessed 2/23/2021).



Definition. The six operations given in Figure 3.1 which can be performed on a knot diagram are the *Reidemeister moves*.

Note. The main result concerning Reidemeister moves is originally due to James Alexander and G. B. Briggs. It is stated as Theorem 3.1.1 below and originally appeared in: J. W. Alexander and G. B. Briggs, On the Types of Knotted Curves, *Annals of Mathematics*, Second Series, **28**, 562–586 (1927); this is available online at Andrew Ranicki's Homepage on the University of Edinburgh Math Department page (accessed 1/30/2021). Recall from Section 2.3. Equivalence of Knots, Deformations that an elementary deformation of a (polygonal) knot is defined in terms of adding a point to a polygonal knot to produce a new (polygonal) knot, with a restriction on the relationship of the original knot and the new knot (in terms of triangles spanned by the points; the restriction insures that the knot does not intersect itself during the deformation). The next result show that equivalence of knots in this sense can be related to a sequence of Reidemiester moves.

Theorem 3.1.1. If two knots (or links) are equivalent, then their diagrams are related by a sequence of Reidemeister moves.

Note. Livingston gives an informal argument for the proof. For a rigorous proof, we refer to Garhard Burde and Heiner Zieschang's *Knots*, Second Revised and Extended Edition, (Berlin, New York: Walter de Gruyter, 2003). See their "Chapter 1. Knots and Isotopies" (pages 2–9). In particular, they define or prove: *ambient isotopy* (Definition 1.2), *equivalent knots* in terms of ambient isotopies (Definition 1.5), Δ *moves* and *combinatorial equivalent* knots (similar to our "elementary deformation" and "knot equivalence," Definitions 1.6 and 1.7), equivalence of ambient-isotopy-knot-equivalence and combinatorial-knot-equivalence (Proposition 1.1), *regular projection* and *knot diagram* (Definition 1.11), and *equivalent knot diagrams/Reidemeister moves* (Definition 1.13). In their terminology, our Theorem 3.1.1 is stated as:

> **Proposition 1.14.** Two knots are equivalent if and only if their diagrams are equivalent.

Burde and Zieschang give a proof on their pages 9 and 10.

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