Section 3.4. Matrices, Labelings, and Determinants

Note. If a knot has a mod p labeling, then at each crossing we require that $2x_i - x_j - x_k \equiv 0 \pmod{p}$ for the arc labels occurring at the crossing (where x_i is the label of the overcrossing arc). To find such a coloring, we need this equation satisfied at each crossing. So we can create a system of equations (modulo p), a solution of which (with a few caveats) gives a mod p labeling. We consider solutions to this (homogeneous) system of equations in terms of the coefficient matrix for the system. We use this matrix to define the determinant and rank of a knot.

Note. We now apply linear algebra to problem of finding a mod p labeling of a graph diagram. We label each arc of a knot diagram with labels x_i where i ranges from 1 to the number of arcs in the diagram. At a crossing, suppose arcs with (not necessarily distinct) labels x_i, x_j, and x_k occur where x_i is the label of the overcrossing arc. Then for this labeling to be a mod p labeling, we need $2x_i - x_j - x_k \equiv 0 \pmod{p}$. We then have such an equation for each crossing, thus producing a homogeneous system of equations; the number of equations is the number of crossings and the number of unknowns is the number of arcs in the diagram. If we find a solution (modulo p) to this system of equations where not all labels are the same, then we have mod p labeling. Since the number of arcs of a knot equals the number of undercrossings (except for the unknot) then the number of arcs equals the number of crossings, so that we have the same number of equations as unknowns in the homogeneous system of equations.

Note. Consider the knot in Figure 3.13.

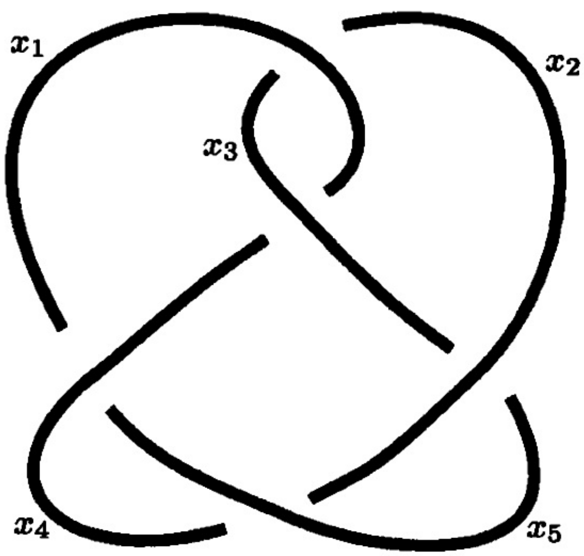

Figure 3.13

There are 5 arcs and 5 arcs, so we get the following 5 equations in 5 unknowns: $2x_1 - x_2 - x_3 \equiv 0 \pmod{p}$, $-x_1 + 2x_3 - x_4 \equiv 0 \pmod{p}$, $-x_1 + 2x_4 - x_5 \equiv 0 \pmod{p}$, $2x_2 - x_3 - x_5 \equiv 0 \pmod{p}$, and $-x_2 - x_4 + 2x_5 \equiv 0 \pmod{p}$. The augmented matrix associated with this system of equations is:

$$\left(\begin{array}{rrrrr|r} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 2 & 0 \end{array}\right).$$

Notice that in Livingston, the augmentation is not given (so the last column is missing in the text book) and the last two rows are interchanged (so our coefficient matrix is row equivalent to the matrix given in Livingston on page 43).

Note. We are interested in a system of equations modulo p (and we look for solutions involving the labels 0 through $p-1$). So we shift the setting some and simply consider the system of equations with the above augmented matrix over the *field* $\mathbb{Z}_p$. The knowledge you gain about systems of equations and matrices in Linear Algebra (MATH 2010), where scalars are real numbers, carry over almost entirely to the setting where scalars are taken from an arbitrary field (in particular, the results about solutions to systems of equations hold over general fields). For a discussion of some of this, see my online notes for Fundamentals of Functional Analysis (MATH 5740) on 5.1. Groups, Fields, and Vector Spaces; notice especially Lemma 5.1.1 which addresses solutions to homogeneous systems of equations. Fields are defined in the Introduction to Modern Algebra (MATH 4127/5127). See my online notes for this class on IV.18. Rings and Fields, and notes for Introduction to Modern Algebra 2 (MATH 4137/5137) on VI.33. Finite Fields. The integers modulo prime p, $\mathbb{Z}_p$, form a field of order p and that's why we require p prime (though not all finite fields are of order p; see the theorem on "The Structure of Finite Fields" in the Finite Fields notes).

Note 3.4.A. Notice that, as illustrated in the example above, we can take all x_i equal and we get a solution to the system of equations (in this case, we have $2x_i - x_j - x_k = 2x_i - x_i - x_i = 0$). However, by the definition of a labeled mod p diagram, we require that at least two labels are distinct. So the most obvious solutions to the system of equations are not permitted. Since the system is homogeneous, then the solution set is the nullspace of the coefficient matrix and hence a linear combination of solutions is again a solution (see, for example, my online notes for Linear Algebra [MATH 2010] on 1.6. Homogeneous Systems, Subspaces and Bases). So if there is a solution where not all x_i are the same, then we can take the solution where all x_i are $-x_n$ and sum these two solutions to get a solution where not all x_i are the same and $x_n = 0$ (of course, we could choose any one of the variables to be 0, it need not be x_n). Hence, a solution to the original system of equations with not all x_i equal corresponds to a nontrivial solution to the system of equations determined by the original matrix with its last column deleted (this new matrix has $n-1$ columns). Now the row rank of a matrix equals the column rank (see Theorem 2.4 in my online Linear Algebra notes on 2.2. The Rank of a Matrix; for a proof, see my online notes for Theory of Matrices [MATH 5090] on 3.3. Matrix Rank and the Inverse of a Full Rank Matrix [see Theorem 3.3.2]). So the matrix with $n-1$ columns can be row reduced to a matrix with the nth row as all 0's. So we can eliminate these last row of 0's and, without loss of generality, consider the resulting $(n-1)\times(n-1)$ matrix. A nontrivial solution of a system of equations with this coefficient matrix exists if and only if a nontrivial solution of the original system of equations exists. This is summarized in the next theorem.

Theorem 3.4.4. There is an $n \times n$ matrix corresponding to a knot diagram with n arcs. Deleting any one column and any one row yields a new matrix. The knot can be labeled mod p if and only if the corresponding set of equations has a nontrivial mod p solution.

Note. Recall that a homogeneous system of equations with a square coefficient matrix has a nontrivial solutions if and only if the coefficient matrix is singular (see, for example, Corollary 2 in my online Linear Algebra notes on 1.6. Homogeneous Systems, Subspaces and Bases). That is, a nontrivial solution exists if and only if the determinant of the coefficient matrix is 0 (see Theorem 4.3 in my online Linear Algebra notes on 4.2. The Determinant of a Square Matrix). These results also hold when the scalars are chosen from the field $\mathbb{Z}_p$ and all arithmetic is done modulo p. This leads us to consider the determinant of the $(n-1) \times (n-1)$ of Note 3.4.A.

Definition. The *determinant of a knot* is the absolute value of the determinant of the associated $(n-1) \times (n-1)$ matrix constructed above.

Definition. The mod p *rank* of a knot is the mod p nullity of the associated $(n-1) \times (n-1)$ matrix constructed above.

Note. The previous two definitions are possibly dependent on the knot diagram used or the ordering of the labels on the arcs and the crossings (that is, the ideas may not be "well-defined"). The next theorem shows that, in fact, they are well-defined.

Theorem 3.4.5. The determinant of a knot and its mod p rank are independent of the choice of diagram and labeling.

Revised: 2/5/2021