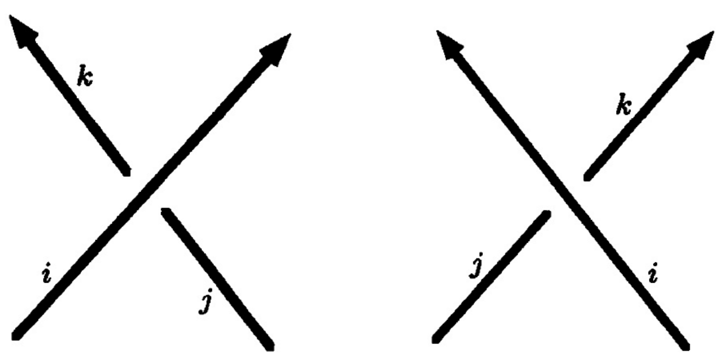
Section 3.5. The Alexander Polynomial

Note. In this section we define a left-handed and a right-handed crossing in an oriented knot diagram. These crossings and a labeling of the arcs are used to create a matrix which is modified and then used to compute the Alexander polynomial using determinants. We give some examples.

Note 3.5.A. We now describe how to compute the Alexander polynomial of a knot, $A_K(t)$. First, pick an oriented diagram for K, number the arcs, and separately number the crossings. Next, define an $n \times n$ matrix where n is the number of crossings (and the number of arcs) in the diagram. We use Figure 3.15 to define a right-handed and left-handed crossing. In Figure 3.15(a), if we curl the fingers of our *right* hand (with thumb pointed out of the page) from the arrow end of the overcrossing arc then our fingers first encounter the arrow end of the undercrossing arc; this is a *left-handed crossing.* In Figure 3.15(b), if we curl the fingers of our *left* hand (with thumb pointed out of the page) from the arrow end of the overcrossing arc then out fingers first encounter the arrow end of the undercrossing arc; this is a *right-handed crossing.* See Exercise 3.2.5 also.



(*a*) *Figure 3.15* (*b*)

If crossing number ℓ is right-handed, then enter a $1-t$ in column i of row ℓ, enter a -1 in column j of row ℓ, and enter a t in column k of row ℓ. If crossing number ℓ is left-handed, then enter a $1-t$ in column i of row ℓ, enter a t in column j of row ℓ, and enter a -1 in column k of row ℓ. All the remaining entries of row ℓ are 0. In the case that any of i, j, or k are equal, the sum of the entries described above are put in the appropriate column. For example, if $j = k$ for some right-handed (left-handed) crossing, then enter $(-1) + (t) = t - 1$ (respectively, $(t) + (-1) = t - 1$) in column $j = k$ of row ℓ; if $i = j = k$, then enter $(1-t)+(-1)+(t) = (1-t)+(t)+(-1) = 0$ in the $i = j = k$ column of row ℓ

Definition. The $(n-1) \times (n-1)$ matrix obtained by removing the last row and column from the $n \times n$ matrix of Note 3.5.A is an *Alexander matrix* of K. The determinant of the Alexander matrix is the *Alexander polynomial* of K. We take the determinant of a 0×0 matrix to be 1, so that the Alexander polynomial of the unknot is 1.

Note. The Alexander polynomial depends on the choice of the diagram for the knot, the choice of the orientation, the choices of the labeling of the crossings, and the choices of the labels of the arcs. However, we can relate the different Alexander polynomials that result as given in the following theorem.

Theorem 3.5.6. If the Alexander polynomial for a knot is computed using two different sets of choices for diagrams and labelings, the two polynomials will differ by a multiple of $\pm t^k$, for some $k \in \mathbb{Z}$.

Note 3.5.B. Notice that by Theorem 3.5.6, there is not a unique Alexander polynomial of a given knot. However, when we say "the Alexander polynomial" of a knot, we refer to the polynomial of minimum degree and positive leading coefficient; these are the polynomials given in Appendix 1. Livingston gives a "Sketch of Proof" of Theorem 3.5.6, but we accept it as given.

Example 3.5.1. Consider the diagram of the trefoil knot of Figure 3.16 with the given orientation and labelings (the arcs are labeled with subscripted x's, but we interpret the labels as the subscripts).

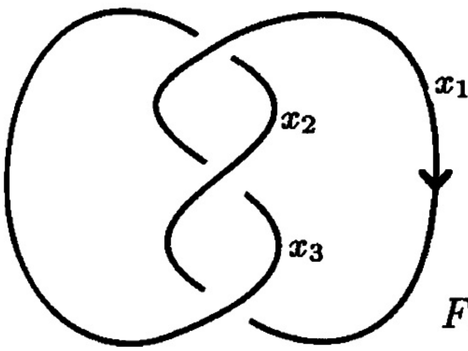


Figure 3.16

We number the crossings from top to bottom as 1, 2, 3. Notice that crossing $\ell = 1$ is a right-hand crossing with $i = 1$, $j = 2$, and $k = 3$ (compare to Figure 3.15(a)). So we take the entries in row $\ell = 1$ as $1 - t$ in column $i = 1$, -1 in column $j = 2$, and t in column $k = 3$. Crossing $\ell = 2$ is a right-hand crossing with $i = 2$, $j = 3$, and $k = 1$ (compare to Figure 3.15(a)). So we take the entries of row $\ell = 2$ as $1 - t$ in column $i = 2$, -1 in column $j = 3$, and t in column $k = 1$. Crossing $\ell = 3$ is a right-hand crossing with $i = 3$, $j = 1$, and $k = 2$ (compare to Figure 3.15(a)). So we take the entries of row $\ell = 3$ as $1-t$ in column $i = 3$, -1 in column $j = 1$, and t in column $k = 2$. The $n \times n = 3 \times 3$ matrix of Note 3.5.A and the

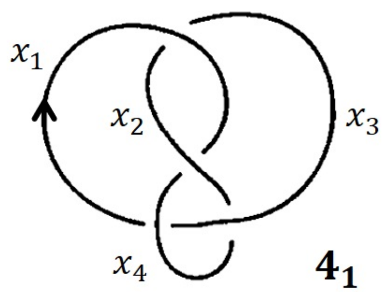
$(n-1)\times(n-1) = 2\times 2$ Alexander matrix are:

$$\begin{pmatrix} 1-t & -1 & t \\ t & 1-t & -1 \\ -1 & t & 1-t \end{pmatrix} \text{ and } \begin{pmatrix} 1-t & -1 \\ t & 1-t \end{pmatrix}.$$

So the Alexander polynomial is

$$\det\begin{pmatrix} 1-t & -1 \\ t & 1-t \end{pmatrix} = (1-t)^2 - (-t) = 1 - 2t + t^2 + t = t^2 - t + 1.$$

Example 3.5.2. Consider the diagram of the knot 4_1 shown here with the given orientation and labelings (the arcs are labeled with subscripted x's, but we interpret the labels as the subscripts).



We number the crossings as 1 (upper-most), 2 (middle), 3 (lower left), and 4 (lower right). Notice that crossing $\ell = 1$ is a right-hand crossing with $i = 1$, $j = 2$, and $k = 3$ (compare to Figure 3.15(a)). So we take the entries in row $\ell = 1$ as $1-t$ in column $i = 1$, -1 in column $j = 2$, and t in column $k = 3$. Crossing $\ell = 2$ is a right-hand crossing with $i = 2$, $j = 1$, and $k = 4$ (compare to Figure 3.15(b)). So we take the entries of row $\ell = 2$ as $1-t$ in column $i = 2$, -1 in column $j = 1$, and t in column $k = 4$. Crossing $\ell = 3$ is a left-hand crossing (arguably) with $i = 4$,

$j = 3$, and $k = 1$ (compare to Figure 3.15(b)). So we take the entries of row $\ell = 3$ as $1 - t$ in column $i = 4$, t in column $j = 3$, and -1 in column $k = 1$. Crossing $\ell = 4$ is a left-hand crossing with $i = 3$, $j = 4$, and $k = 2$ (compare to Figure 3.15(b)). So we take the entries of row $\ell = 4$ as $1 - t$ in column $i = 3$, t in column $j = 4$, and -1 in column $k = 2$. The $n \times n = 4 \times 4$ matrix of Note 3.5.A and the $(n-1) \times (n-1) = 3 \times 3$ Alexander matrix are:

$$\begin{pmatrix} 1-t & -1 & t & 0 \\ -1 & 1-t & 0 & t \\ -1 & 0 & t & 1-t \\ 0 & -1 & 1-t & t \end{pmatrix} \text{ and } \begin{pmatrix} 1-t & -1 & t \\ -1 & 1-t & 0 \\ -1 & 0 & t \end{pmatrix}.$$

Using the cofactor expansion for determinants, we have "an" Alexander polynomial

$$\det \begin{pmatrix} 1-t & -1 & t \\ -1 & 1-t & 0 \\ -1 & 0 & t \end{pmatrix} = (1-t)\det \begin{pmatrix} 1-t & 0 \\ 0 & t \end{pmatrix}$$

$$-(-1)\det \begin{pmatrix} -1 & t \\ 0 & t \end{pmatrix} + (-1)\det \begin{pmatrix} -1 & t \\ 1-t & 0 \end{pmatrix}$$

$$= (1-t)(1-t)(t) - (-1)(-1)(t) + (-1)(-t)(1-t) = (t - 2t^2 + t^3) - (t) + (t - t^2)$$

$$= t^3 - 3t^2 + t = t(t^2 - 3t + 1).$$

Ignoring the factor of t as described in Notes 3.5.B, we get "the" Alexander polynomial of knot $K = 4_1$ as $A_K(t) = t^2 - 3t + 1$, in agreement with Appendix 2.

Note. We'll see in Chapter 6, "Geometry, Algebra, and the Alexander Polynomial," that the Alexander polynomial is symmetric (see Corollary 6.2.2 and Appendix 2 which lists the Alexander polynomials for the knots of Appendix 1); that is, for a given knot K, $A_K(t) = \sum_{i=0}^{n} a_i t^i$ where $a_i = a_{n-i}$ so that the coefficients are symmetric (not to be confused with a "symmetric function" which you may encounter in algebra; see my online notes for Introduction to Modern Algebra 2 on Section X.54. Illustrations of Galois Theory). Though the Alexander polynomial is not strictly speaking a knot invariant (see Theorem 3.5.6 and Note 3.5.B), we can conclude that if two knots have Alexander polynomials that do not differ by a multiple of $\pm t^k$ for some $k \in \mathbb{Z}$ then the knots are not equivalent. Livingston claims that the Alexander polynomial of the $(2, n)$-torus knot is $(t^n + 1)/(t + 1)$. These polynomials differ for each n so that we have an infinite collection of nonequivalent knots in the collection of $(2, n)$-torus knots.

Revised: 2/5/2021