Section 4.1. Surfaces and Homeomorphisms

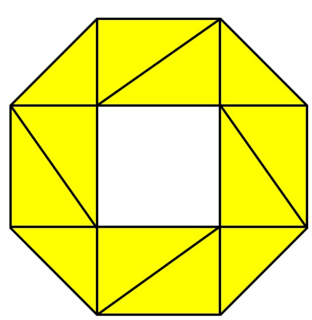
Note. In this section we deal with general smooth surfaces as polyhedral surfaces, define orientable surfaces, and define a surface homeomorphism.

Note. Any three noncollinear points in 3-space, p_1, p_2, and p_3, form the vertices of a unique triangle. Treating points as vectors in 3-space (and, UGH!) then we can treat the set of points in the triangle are

$$\{xp_1 + yp_2 + zp_3 \mid x + y + z = 1,\ x, y, z \geq 0\}.$$

The union of a finite collection of triangles is a *polyhedral surface* if (1) each pair of triangles is either disjoint or their intersection is a common edge or common vertex, (2) at most two triangles share a common edge, and (3) the union of the edges that are contained in exactly one triangle is a disjoint collection of simple polygonal curves, called the *boundary* of the surface.

Note. An example of a polyhedral surface is the polyhedral annulus given here:



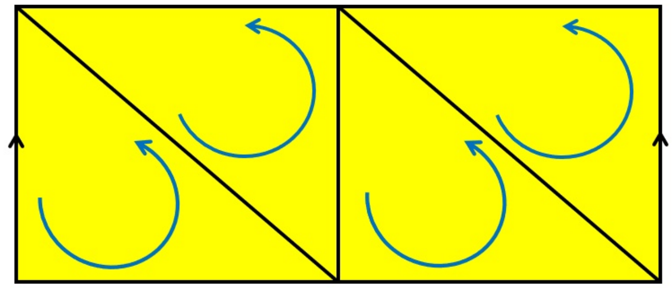
A Polyhedral Annulus. The boundary consists of a square and an octagon.

Note. We could define a surface in terms of a differentiable structure (as we could have done for knots). However, similar to knots, we can use polyhedra (instead of polygons) for a simpler approach. See my online notes for Differential Geometry (MATH 5310) on 1.9. Manifolds where a "smooth" surface is defined. Some supplemental online notes for Differential Geometry (based on Stephen Hawkings and G. F. T. Ellis' *The Large Scale Structure of Space-Time*, Cambridge University Press (1973)) on 2.1. Manifolds is also available in which a general manifold n-dimensional manifold is defined with a certain level of smoothness. This source also considers a manifold with a boundary. In this language, we consider smooth 2-dimensional manifolds with a boundary here.

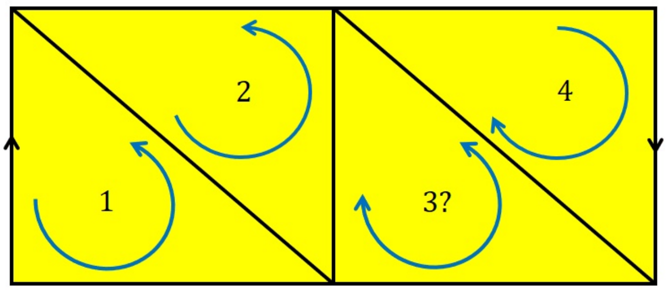
Note. In drawings, surfaces will be drawn smoothly. Such a surface can be approximated by a polyhedral surface, called a *triangulation* of the surface.

Definition. A polyhedral surface is *orientable* if it is possible to orient the boundary of each of its constituent triangles in such a way that when two triangles meet along an edge, the two induced orientations of that edge run in opposite directions.

Note. The following figure shows that a cylinder is an orientable surface. Consider the left-hand and right-hand edges as coinciding, so that the upper edge and the lower edge form the boundary of the cylinder. A triangulation is given and the directed arc in each triangle indicates how the boundary edges of that triangle are oriented. Notice that triangles sharing an edge give that shared edge opposite orientations, as requires by the above orientation.



Note. A Möbius strip is not orientable. Consider the following figure. Consider the left-hand and right-hand edges as coinciding, but with a twist (so that the upper left points coincide with the lower right points). Notice that the single boundary includes both the upper edge and the lower edge of the rectangle. A triangulation is given. Without loss of generality, give triangle 1 a counterclockwise orientation as given. Notice that the right-hand edge of triangle 1 will then be oriented in the opposite direction as the arrow on the left-hand edge of the rectangle. So triangle 4 must be given the clockwise orientation that gives the right-hand edge of triangle 4 an orientation in the same direction as the right-hand edge of the rectangle. Triangle 2 will have to be given a counterclockwise orientation because of the edge it shares with triangle 1. But then triangle 3 needs a counterclockwise orientation because of the edge it shares with triangle 1, and it needs a clockwise orientation because of the edge it shares with triangle 4. So no required orientations of the edges of the triangles exists and this surface is not orientable. Livingston says (see page 58): "The intuitive approach to orientability states that a surface is orientable if it is two-sided."



Note. We want to define an idea of equivalence of surfaces. The "intrinsic" properties of a surface are those properties that an inhabitant can detect, such as the number of boundary components and orientability. Two surfaces which share the same intrinsic properties are "homeomorphic." We now formalize this.

Definition. Surfaces F and G is 3-space are *homeomorphic* is there is a continuous function with domain F and range G which is both one-to-one and onto, and a continuous inverse.

Note. Livingston omits the condition "and a continuous inverse" in his definition of a homeomorphism. Our approach is standard, as can be seen in my online notes for Introduction to Topology (MATH 4357/5357) on Section 18. Continuous Functions, which gives a definition of homeomorphism as we have used.

Note. Any triangle can be subdivided to introduce more more triangles, more vertices, and more edges, as illustrated in Figure 4.3. These subdivisions are called "finer" triangulations.

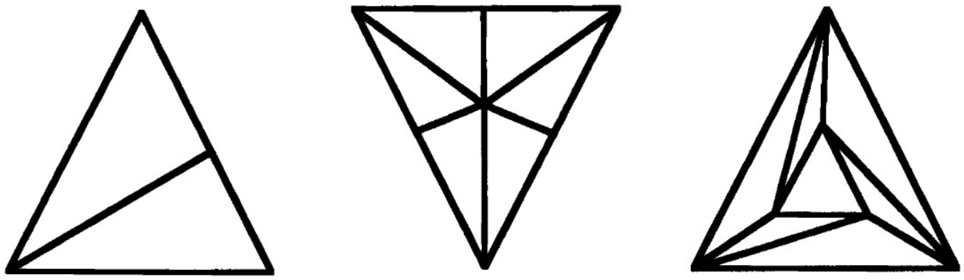


Figure 4.3

Definition. Polyhedral surfaces are *homeomorphic* if, after some subdivision of the triangulations of each, there is a continuous bijection with a continuous inverse between the vertices of the surfaces such that when three vertices in one surface bound a triangle the corresponding vertices in the second surface also bound a triangle.

Note. Consider the surfaces in Figure 4.4. These surfaces are homeomorphic because they have the same intrinsic properties. The knottedness of the surface on the left is not present in the surface on the right, but knottedness is not an intrinsic property of the surface. Instead, it is an "extrinsic" property that depends on how the surface is embedded in 3-space.

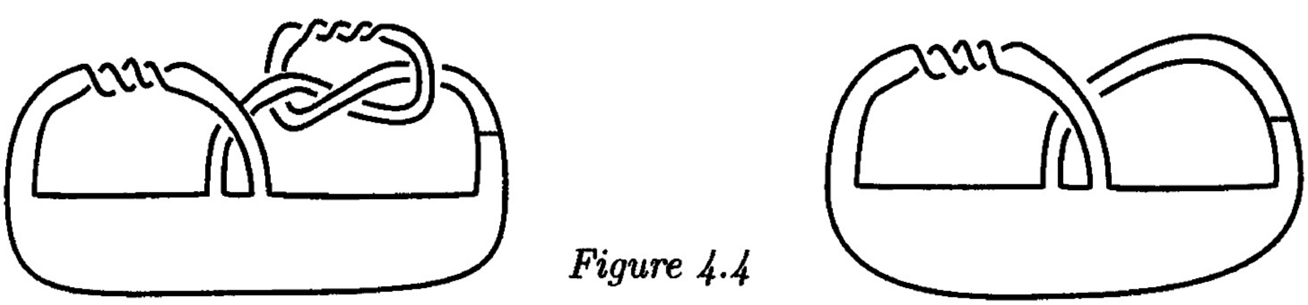


Figure 4.4

A homeomorphism between the surfaces is given by a map that cuts the left surface along the dotted line, unknots and untwists the band, and then reattaches it. This mapping is one-to-one and onto, and continuity follows (informally) because points close together on one surface are mapped to points close together on the other surface.

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