Section 4.2. The Classification of Surfaces

Note. In this section we discuss the genus of a surface and several versions of the classification of surfaces. We define the Euler characteristic of a surface and give several results concerning the genus and Euler characteristic of unions of surfaces. We state two classification surfaces which relate to homeomorphisms of surfaces with a boundary.

Note. In Figure 4.6, we consider three orientable surfaces without boundaries. With such surfaces is associated a number called the "genus" of the surface (the genus of an nonorientable surface can also be defined). Crudely put, the genus of a surface is the number of "holes" in the surface, so that Figure 4.6 gives surfaces of genus 0, 1, and 3.

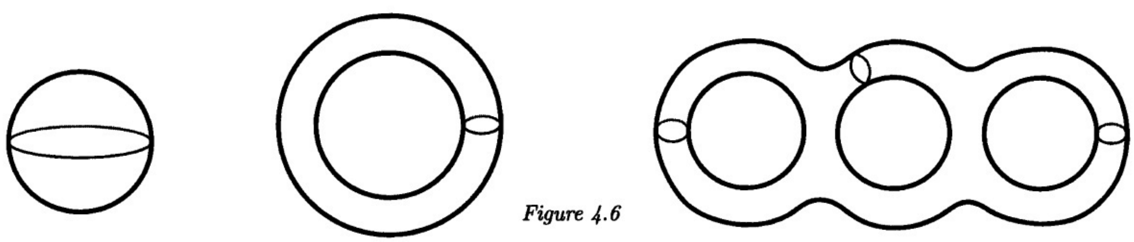

Figure 4.6

Note. Surfaces are classified, up to homeomorphism, by The Classification Theorem. Livingston paraphrases this as (see page 62): "A theorem, called the *classification of surfaces*, implies that connected oriented surfaces *without boundary* are homeomorphic if and only if they have the same genus." This is more precisely stated in James Munkres' *Topology*, Second Edition, Upper Saddle River, NJ: Prentice Hall (2000) as:

Theorem 77.5. The Classification Theorem.

Let X be the quotient space obtained from a polygonal region in the plane by pasting its edges together in pairs. Then X is homeomorphic to either S^2, to the n-fold torus T, or to the m-fold projective plane P_m.

The surface S^2 is the sphere in $\mathbb{R}^3$ (of genus 0), the n-fold torus is an orientable surface of genus n, and the m-fold projective plane is a nonorientable surface of genus m. Another statement of The Classification Theorem is given in George Francis and Jeffrey Weeks' "Conway's ZIP Proof," *American Mathematical Monthly*, **106**(5), 393–399 (May 1999); a copy is available online at "Conway's ZIP Proof" (accessed 2/12/2021). In this paper,

Classification Theorem (Conway's Version).

Every connected closed surface is homeomorphic to either a sphere with crosscaps or a sphere with handles.

A sphere with handles is an orientable surface where the genus equals the number of handles, and a sphere with crosscaps is a nonorientable surface. Francis and Weeks' paper is reprinted in Jeffrey Weeks' *The Shape of Space*, Second Edition, Monographs and Textbooks in Pure and Applied Mathematics #249, NY: Marcel Dekker (2002), Appendix C. This is an excellent reference on the geometry of surfaces and 3-manifolds.

Note. My online notes for Graph Theory 2 (MATH 5450) on Section 10.6. Surface Embeddings of Graphs give several of the details concerning handles, crosscaps, and the Euler characteristic of a surface. We now state a definition of the Euler characteristic of a surface in terms of a triangulation.

Definition. If a polyhedral surface S is triangulated with F triangles, and there are a total of E edges and V vertices in the triangulation, then the *Euler characteristic* is given by $\chi(S) = F - E + V$.

Note. The fact that the Euler characteristic is unaffected by the choice of the triangulation of the surface follows from a proof in Section 3.1. "Classification of Surface" in Bojan Mohar and Carsten Thomassen's *Graphs on Surfaces*, Baltimore: Johns Hopkins University Press (2000). I have online notes for some of this book at Topological Graph Theory notes.

Note. The octahedron in Figure 4.7 satisfies $F = 8$, $E = 12$, and $V = 6$, so that $\chi(S) = F - E + V = 8 - 12 + 6 = 2$.

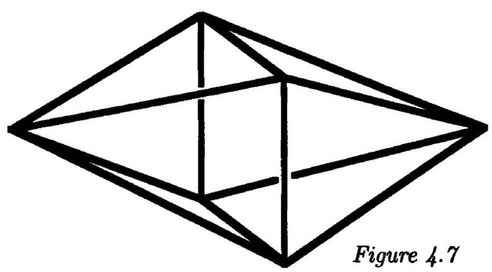

Figure 4.7

Definition. The *genus* of a connected orientable surface S is given by

$$g(S) = \frac{2 - \chi(S) - B}{2}$$

where B is the number of boundary components of the surface.

Note. In the following we consider the Euler characteristic of the union of two surfaces, where we "glue" part of a boundary of one surface to part of a boundary of the other surface. We use the term "arc" to indicate pieces of the boundaries which are glued together (so an arc is a maximal connected part of a boundary that is glued to part of the boundary of another surface).

Theorem 4.2.1. If two surfaces intersect in a collection of arcs contained in their boundary, the Euler characteristic of the union is the sum of their individual Euler characteristics minus the number of (common) arcs of intersection.

Example 4.2.1. Many of the surfaces of interest to us are formed from disks with twisted bands added. We recall Figure 4.4 from the previous section here.

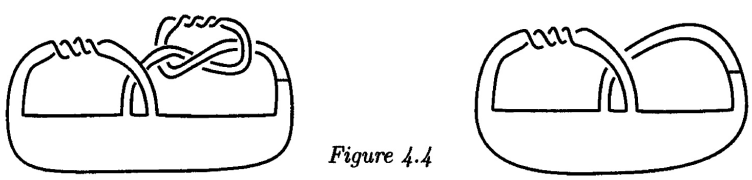

Figure 4.4

Now the disk has Euler characteristic 1 (think of it as a single triangle with $F = 1$, $E = 3$, and $V = 3$ so that $\chi = F - E + V = 1 - 3 + 3 = -1$). Similarly, a "band" is just an elongated disk and so also has Euler characteristic 1. So if we attach several bands to a disk (making 2 attachments per band; so the number of common arcs is twice the number of bands), by Theorem 4.2.1 we get a surface of Euler characteristic:

$$(1 + \#\text{bands}) - 2(\#\text{bands}) = 1 - \#\text{bands}.$$

So both surfaces in Figure 4.4 have Euler characteristic $(1+(2)) - 2(2) = 1-(2) = -1$.

Note. The next result is similar to Theorem 4.2.1, but deals with the genus of a union of two surfaces. The proof is based on expressing the Euler characteristic in terms of the genus (using the definition of genus) and using Theorem 4.2.1. Details are to be given in Exercise 4.2.3.

Corollary 4.2.2. If two connected orientable surfaces intersect in a single arc contained in each of their boundaries, the genus of the union of the two surfaces is the sum of the genus of each.

Note. The next result has a proof similar to that of Theorem 4.2.1. It involves the genus of a surface that results from attaching bands to a collection of disks.

Theorem 4.2.3. If a connected orientable surface is formed by attaching bands to a collection of disks, then the genus of the resulting surface is given by

$$(2 - \#\text{disks} + \#\text{bands} - \#\text{boundary components})/2.$$

Note. The next theorem will be useful when we consider "surgery" on surfaces in Section 4.4.

Theorem 4.2.4. If two surfaces intersect in a collection of circles contained in the boundary of each, the Euler characteristic of their union is the sum of their Euler characteristics.

Note. We are mostly interested in surfaces with boundaries, since the boundaries will form knots in our future studies. So we give two classification results for such surfaces. The first classification theorem shows the fundamental importance of attaching bands to disks. The second classification theorem classifies homeomorphic surfaces with boundaries in terms of numbers of bands, numbers of boundary components, and orientability.

Theorem 4.2.5. Classification I. Every connected surface with boundary is homeomorphic to a surface constructed by attaching bands to a disk.

Theorem 4.2.6. Classification II. Two disks with bands attached are homeomorphic if and only if the following three conditions are met:

(1) they have the same number of bands,

(2) they have the same number of boundary components, and

(3) both are orientable or both are nonorientable.

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