# **Section 4.3.** Seifert Surfaces and the Genus of a Knot

**Note.** We now associate a knot (or, more generally, a link) with a surface, called the Seifert surface. The existence of such a surface for any link was first proved in F. Frankl and L. Pontrjagin's "Ein Knotensatz mit Anwendung auf die Dimensionstheorie [A Set of Nodes with Application to Dimension Theory]," *Mathematische Annalen*, **102**(1), 785-789 (1930). A different proof was published by Seifert using what is now called the "Seifert algorithm." This appeared in Herbert Seifert, (1934). "Über das Geschlecht von Knoten [About the Sex of Knots]," *Mathematische Annalen*, **110**(1), 571-592 (1934). (Some of this history is based on the Wikipedia page on Seifert Surfaces.)

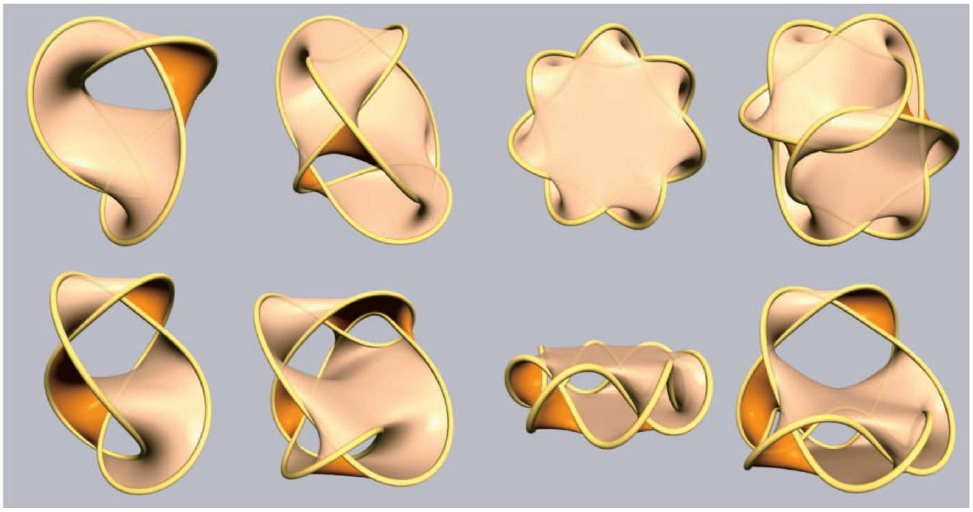
Karl Herbert Seifert, May 27, 1907–October 1, 1996

From the MacTutor History of Mathematics Archive page on Karl Seifert.

**Theorem 4.3.7.** Every knot is the boundary of an orientable surface.

**Definition.** An orientable surface with a given knot as its boundary is a *Seifert surface* for the knot.

**Note.** Here is an image of four Seifert surfaces of knots (two perspectives of each are given).



From the ThatsMaths page on Seifert Surfaces for Knots and Links.

**Definition.** The *genus* of a knot is the minimum possible genus of a Seifert surface for the knot.

**Note.** In Figure 4.1 we see that the trefoil knot bounds a surface of Euler characteristic $\chi(S) = -1$ (by Example 4.2.1) and, since the surface has $B = 1$ boundary component, the genus of the surface is (by definition) $g(S) = (2 - \chi(S) - B)/2 = 2 - (-1) - 1)/2 = 1$. Notice that the trefoil knot cannot have a Seifert surface of genus 0 (a disk), because this it would be unknotted. So the genus of a trefoil knot is 1.

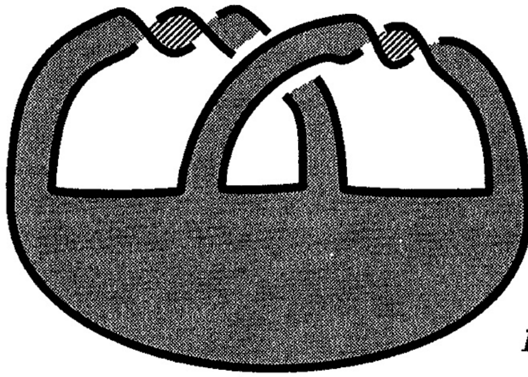


*Figure 4.1*

**Note.** As Livingston comments (see page 72), the Seifert algorithm may not produce a minimal genus surface. So finding the genus of a knot can be difficult.

*Revised: 2/17/2021*