Section 4.4. Surgery on Surfaces

Note. We define "puncturing" a surface and surgery on a surface. We show how surgery affects the genus of a surface. From now on in these notes, **we assume all surfaces are orientable**.

Note. If two surfaces intersect along intervals or circles contained in their boundaries, then the union of the surfaces is again a surface. In Theorem 4.2.1 we addressed how the Euler characteristic of the union of such surfaces is related to the Euler characteristic of the constituent surfaces in the case of shared arcs. In Theorem 4.2.4 we addressed how the Euler characteristic of the union of such surfaces is related to the Euler characteristic of the constituent surfaces in the case of shared circles. Also notice that if one surface is a subset of another surface and the two surfaces have disjoint boundaries, then removal of the interior of the subset surface results in a new surface. For example, removing a disk interior to a large disk results in an annulus.

Definition. Let D be a disk which is a subset of some surface S, where the boundary of D and the boundary of S are disjoint. The process of removing the interior of D from S is called *puncturing* the surface S.

Note. We now describe some examples of "surgery" on a surface. Consider the case of F as a surface in 3-space and D a disk in 3-space such that the interior of D is disjoint from surface F and the boundary of D lies in the interior of F. See Figure 4.15.

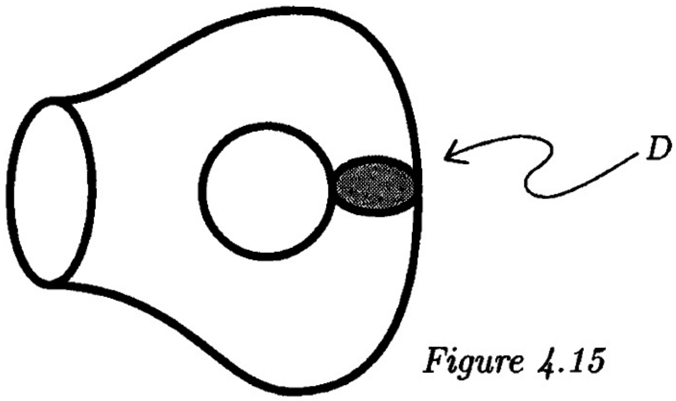

Figure 4.15

Now remove a strip (that is, an annulus) on F which is centered on the boundary of D in F. The new surface that results has two new boundaries that are not boundaries of the original surface F (and both are circles). Attach two disks parallel to disk D to the new surface by joining the boundary of the new disks to the new boundary circles of the new surface. See Figure 4.16 (where the two ovals on the left of the figure indicate the two new disks).

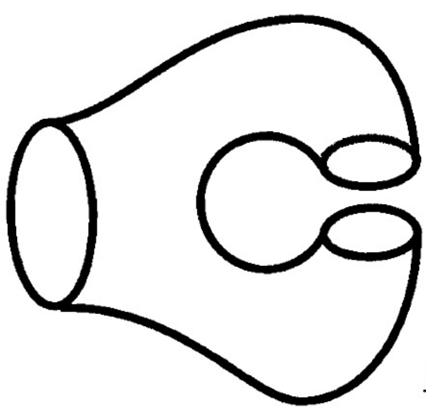

Figure 4.16

Definition. The procedure just described of removing the boundary of disk D from surface F and adding two new disks as described is referred to as performing *surgery* on F along D. When surgery on surface F with one component results in a surface with two components, then the boundary of disk D is called *separating*.

Note. Consider the surface F of a cylinder (including the flat surfaces on the ends, say). Let D be a circular disk with its boundary lying on the F (but not on the ends of the cylinder); notice that D then shares no boundary points with F. Then surgery on F along D results in two surfaces so this is an example of a separating disk. This is like cutting a cucumber in half along a plane perpendicular to its axis!

Note. The next result shows how surgery affects the genus of a surface.

Theorem 4.4.8. If surgery on a connected orientable surface F results in a connected surface F' then $\text{genus}(F') = \text{genus}(F) - 1$. If surgery results in a surface with two components, F' and F'', then $\text{genus}(F) = \text{genus}(F') + \text{genus}(F'')$.

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