# Section 4.5. Connected Sums of Knots and Prime Decompositions

**Note.** In this section we informally describe the connected sum of two knots. We define a prime knot and state the "Prime Decomposition of Knots" theorem. We also prove that the genus of the connected sum of two knots is the sum of the genuses of the knots in the sum (see Theorem 4.5.10).

**Note.** We give a picture-motivated definition of the connected sum of two knots. In Figure 4.17, a sphere in 3-space intersects a knot $K$ in exactly two points. The sphere splits the knot into two arcs, one inside the sphere and the other outside. The endpoints of either arc can be joined by an arc lying on the sphere, producing two knots which we denote as $K_1$ and $K_2$.

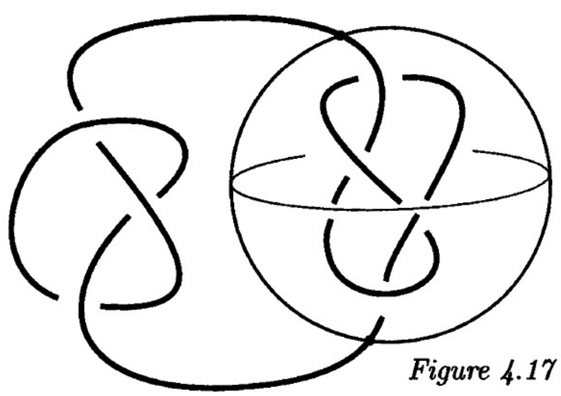
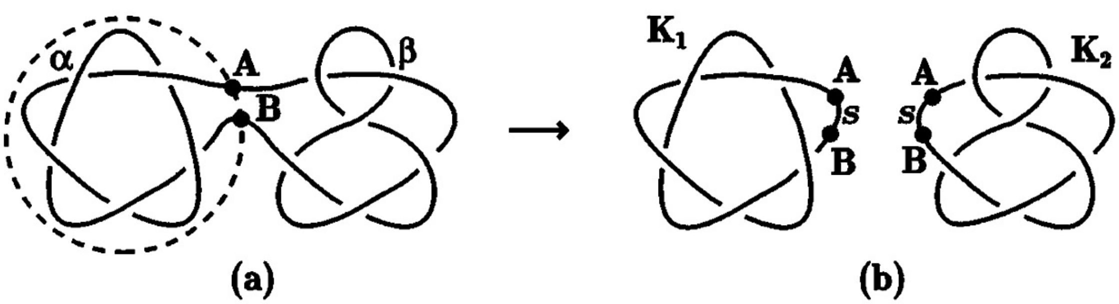


*Figure 4.17*

**Definition.** The situation illustrated in Figure 4.17 is an example of expressing knot $K$ as the *connected sum* of $K_1$ and $K_2$, denoted $K = K_1 \# K_2$.

**Note.** Given knots $K_1$ and $K_2$ it is easy enough to construct *a graph* $K$ such that $K = K_1\#K_2$ (by "joining" the diagrams, for example). "Surprisingly," $K$ is not uniquely determined by $K_1$ and $K_2$. One way to think of the difficulties is that in computing $K_1\#K_2$, we choose some arc of $K_1$ and some arc of $K_2$ and then join $K_1$ and $K_2$ using these arcs. But there are multiple from which to choose and then when joining the graphs, there are two ways to attach the arcs from $K_1$ and $K_2$ together. An example if given in the figure below from *Knot Theory and Its Applications*, by Kunio Murasugi (translated by Bohdan Kurpita), Boston: Birkhauser (1996, originally published in Japanese in 1993).




Livingston states (page 77) that "if $K_1$ and $K_2$ are oriented knots it is possible to find a unique oriented knot $K$ such that $K = K_1\#K_2$ as oriented knots." This is illustrated in the figure below, which is Figure 1.6 from W. B. Raymond Lickorish's *An Introduction to Knot Theory*, Graduate Texts in Mathematics #175, NY: Springer (1997).

We take *connected sum* as sufficiently "carefully defined" and now use the idea.

**Definition.** A knot is *prime* if for any decomposition as a connected sum, one of the factors is unknotted.

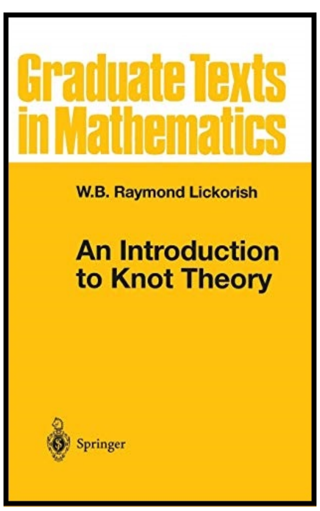
**Theorem 4.5.9. Prime Decomposition Theorem.**
Every knot can be decomposed as the connected sum of nontrivial prime knots. If $K = K_1 \# K_2 \# \cdots \# K_n$, and $K = J_1 \# J_2 \# \cdots \# J_m$, with each $K_i$ and $J_i$ nontrivial prime knots, then $m = n$ and, after reordering, each $K_i$ is equivalent to $J_i$.

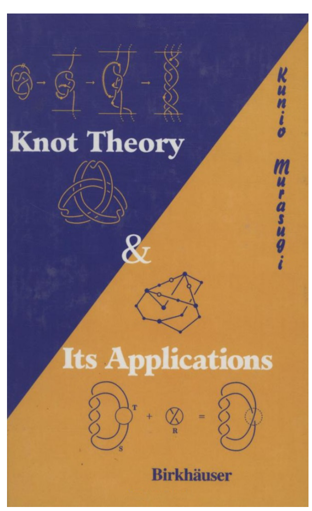
**Note.** Livingston argues that an inductive proof of the existence part of the Prime Decomposition Theorem can be given based on Theorem 4.5.10, "Additivity of Knot Genus" (see page 78). But the uniqueness part of the Prime Decomposition Theorem is not addressed.

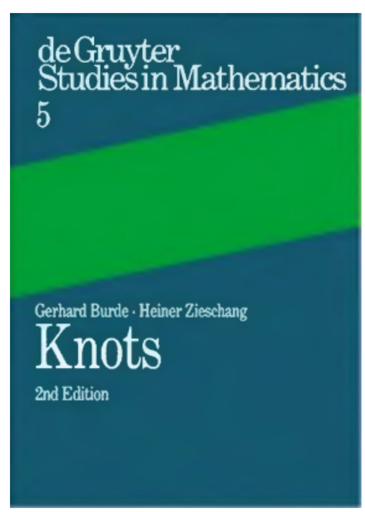
**Note.** The Prime Decomposition Theorem was first proved by Horst Schubert (June 11, 1919–2001). See: Horst Schubert, "Die eindeutige Zerlegbarkeit eines Knotens in Primknoten [The Unique Decomposability of a Knot into Prime Knots]," *Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse*, **3**, 57–104 (1949). A rigorous proof in English can be found in: Gerhard Burde and Heiner Zieschang, *Knots* Second Revised and Extended Edition, de Guyter Studies in Mathematics 5, Berlin: Walter de Gruyter (2003). See "Chapter 7. Factorization of Knots, Section B. Uniqueness of the Decomposition into Prime Knots: Proof" (page 98, subsection 7.18; the theorem is stated as Theorem "7.12. The Unique Prime Decomposition of Knots" on page

96). A largely self-contained paper based on Burde and Zieschang's proof is given in Michael Sullivan's "Knot Factoring," *American Mathematical Monthly*, **107**(4), 297–315 (2000) (accessed 2/16/2021). A proof is also given in Lickorish's book mentioned above (*An Introduction to Knot Theory*, Graduate Texts in Mathematics #175, NY: Springer (1997)); see Chapter 2, "Seifert Surfaces and Knot Factorization," and Theorem 2.12.

**Note.** We take this opportunity to mention that Livingston's book is appropriate for an undergraduate Independent Study (MATH 4900) or as a supplement for Introduction to Topology (MATH 4357/5357). For a purely graduate level class on knot theory, the following references which we have mentioned are appropriate: **(1)** W. B. Raymond Lickorish, *An Introduction to Knot Theory*, Graduate Texts in Mathematics #175, NY: Springer (1997), **(2)** Kunio Murasugi, *Knot Theory and Its Applications* (translated by Bohdan Kurpita), Boston: Birkhauser (1996, originally published in Japanese in 1993), and **(3)** Gerhard Burde and Heiner Zieschang, *Knots* Second Revised and Extended Edition, de Guyter Studies in Mathematics 5, Berlin: Walter de Gruyter (2003).



I have online notes for the Lickorish book (in preparation), and the Burde and Zieschang book went into a third edition in 2013.

**Theorem 4.5.10. Additivity of Knot Genus.**
If $K = K_1 \# K_2$ then $\text{genus}(K) = \text{genus}(K_1) + \text{genus}(K_2)$.

**Note.** We now use Theorem 4.5.10 to prove the existence part of the Prime Decomposition Theorem: Every knot can be decomposed as the connected sum of nontrivial prime knots.

**Note.** The proof technique used in the proof of Theorem 4.5.10 is often called "cut-and-paste" (for obvious reasons) or "inner-most circle argument." The uniqueness part of the Prime Decomposition Theorem (Theorem 4.5.9) also follows from a cut-and-paste argument. Since the genus of a knot is nonnegative, then Theorem 4.5.10 ("Additivity of Knot Genus") implies the following.

**Corollary 4.5.11.** If $K$ is nontrivial, there does not exist a knot $J$ such that $K \# J$ it trivial.

*Revised: 2/16/2021*