Chapter 5. Algebraic Techniques

Section 5.1. Symmetric Groups

Note. In this section we define a "group" in general, but concentrate on the symmetry groups S_n. For our applications to knot theory, we only need an introduction to symmetry groups. As a warning, when using the cycle notation in S_n we multiply cycles by **reading left-to-right**, instead of the standard right-to-left way of multiplying cycles encountered in Introduction to Modern Algebra (MATH 4127/5127).

Definition. A *permutation* on a set T is a one-to-one function (i.e., a bijection) from set T to itself.

Note. If f and g are permutations on set T, then the composition $f \circ g$ (defined **here** as $f \circ g(i) = g(f(i))$) is also a permutation on set T. This follows from the fact that a composition of bijections is again a bijection; see Theorem 1-2 in my online notes for Analysis 1 (MATH 4217/5217) on Section 1.1. Sets and Functions. Notice that the composition of two functions is reversed from the usual definition. This "is fairly standard in knot theory" according to Livingston (see page 84). We will only deal with permutations on finite sets and will introduce a notation next that will differ from the notation you likely saw in your Introduction to Modern Algebra (MATH 4127/5127) class.

Definition. For $T = \{1, 2, , \ldots, n\}$, the set S_n of all permutations on set T is called the *symmetric group* on n symbols, denoted S_n.

Note. Recall from Introduction to Modern Algebra (MATH 4127/5127) that a *group* G is set of elements with a binary operation $*$ (a mapping of $G \times G$ into G; we denote the element of G associated with $(a, b) \in G$ under $*$ as $a * b$) such that (1) for all $a, b, c \in G$ we have $(a * b) * c = a * (b * c)$ (associativity of $*$), (2) there is $e \in G$ such that $e * a = a * e = a$ for all $a \in G$ (e is an *identity* in G), and (3) for all $a \in G$, there is $a' \in G$ such that $a * a' = a' * a = e$ (a' is an *inverse* of a). So S_n is in fact a group where the binary operation is function composition (function composition is associative giving property (1); properties (2) and (3) are clear).

Note. Livingston concentrates on set S_5 of all permutations on the set $\{1, 2, 3, 4, 5\}$. Notice that there are $5! = 120$ permutations on these symbols. One of these is the function f such that $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 5$, and $f(5) = 1$; we denote this permutation in *cyclic notation* as $(1, 2, 3, 4, 5)$. This is a "5-cycle" and its "length" is 5.

Note. The permutation $g \in S_5$ such that $g(1) = 3$, $g(3) = 2$, $g(2) = 1$, $g(4) = 5$, and $g(5) = 4$ can be written as a "product" of two cycles as $(1, 3, 2)(4, 5)$ (this is Example 5.1.3). More generally, we multiply cycles by reading **left-to-right** (in contrast to the way you likely dealt with cycle multiplication in Introduction to Modern Algebra; see my online notes on Section II.8. Groups of Permutations for more information on permutation groups in general, and see Section II.9. Orbits, Cycles, and the Alternating Groups for more on the cycle notation). So if we consider the product $(1,3,2)(2,3)(1,5,4)$, then (reading from left-to-right) we have that the first cycle maps 1 to 3, and then the second cycle maps 3 to 2, so that the product maps 1 to 2. For 2, the first cycle maps 2 to 1, and then the third cycle maps 1 to 5, so that the product maps 2 to 5. Similarly, 3 is mapped to 2 by the first cycle, and 2 is mapped to 3 by the third cycle, so that the product maps 3 to 3. Element 4 is only present in the third cycle where it is mapped to 1, and element 5 is only present in the third cycle where it is mapped to 4. Therefore, the product can be written as $(1,3,2)(2,3)(1,5,4) = (1,2,5,4)(3)$. Notice that the two cycles $(1,2,5,4)$ and (3) are disjoint. It is also standard to exclude the cycles of length one so that we might write $(1,3,2)(2,3)(1,5,4) = (1,2,5,4)$ (this is Example 5.1.4). Example 5.1.5 claims

$$(1,3,4)(1,4,5)(2,3)(1,3,2,5,4)(1,4)(2,5,3) = (1,3)(2,5).$$

You should convince yourself that this is correct.

Definition. A set of permutations $\{g_1, g_2, \ldots, g_k\}$ *generates* the symmetric group if every element in the group can be written as a product of elements from the set, with possible repetitions and use of their inverses. A *transposition* is a cycle of length 2.

Note. In Exercise 5.1.7(d) it it to be shown that the set of transpositions $\{(1,2)$, $(2,3)$, $(3,4)$, $(4,5)$, $\ldots$, $(n-1,n)\}$ are a generating set of the symmetric group S_n.

Note. For what follows, we do not need to explore group theory in great depth and we will get by just with a bit of knowledge of the symmetric groups S_n. We will have results that hold for general groups, but examples will be based on symmetric groups.

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