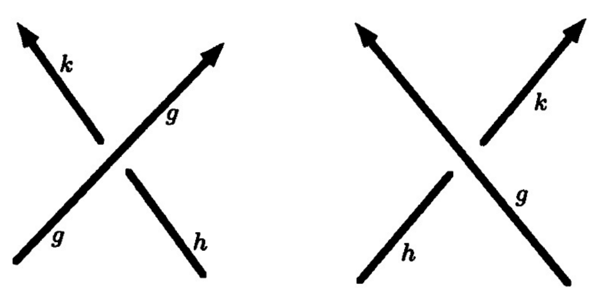
# **Section 5.2.** Knots and Groups

**Note.** In this section we define a process of labeling the arcs of a oriented knot diagram with group elements. We prove that if one oriented diagram of a knot can be so labeled, then any other oriented diagram of the same knot can be also be labeled with elements from the same group.

**Definition.** A *labeling of an oriented knot diagram with elements of a group* consists of assigning an element of the group to each arc of the diagram in such a way that the following two conditions hold.

**(1)** *Consistency*: At each crossing of the diagram three arcs appear, each of which is labeled with an element from the group. If the labels are group elements $g$, $h$, and $k$, then in the case of a right-handed crossing the labels satisfy $gkg^{-1} = h$, and in the case of a left-handed crossing the labels satisfy $ghg^{-1} = k$. See Figure 5.1(a) and Figure 5.1(b), respectively.

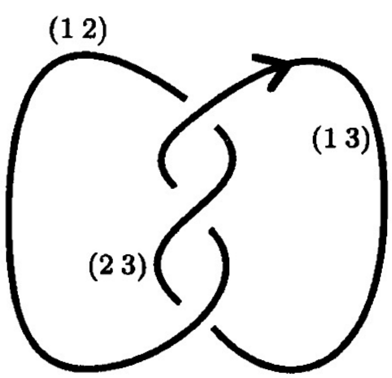



(a) $gkg^{-1} = h$ (b) $ghg^{-1} = k$

*Figure 5.1*

**(2)** *Generation*: The labels must form a set that generates the group. That is, every element of the group can be written as a product of the elements that appear as labels, along with their inverses.

**Example.** In Figure 5.2, two labelings of the trefoil knot are given. In Figure 5.2(a), a labeling of the arcs is given by elements of the group $S_3$. Since every transposition in $S_3$ is an arc label, then the arc labels generate $S_3$ by Exercise 5.1.7(d) and we have "Generation" satisfied. A transposition is its own inverse, so we can check the three crossings. The upper crossing is right-handed in which case we have $g = (1,3)$, $k = (1,2)$, and $h = (2,3)$ (notice the error in the book on the location of label $(2,3)$), so that $gkg^{-1} = (1,3)(1,2)(1,3) = (1)(2,3) = (2,3) = h$ as needed. The middle crossing is right-handed in which case we have $g = (2,3)$, $k = (1,3)$, and $h = (1,2)$, so that $gkg^{-1} = (2,3)(1,3)(2,3) = (1,2)(3) = (1,2) = h$ as needed. The lower crossing is right-handed in which case we have $g = (1,2)$, $k = (2,3)$, and $h = (1,3)$, so that $gkg^{-1} = (1,2)(2,3)(1,2) = (1,3)(2) = (1,3) = h$ as needed. Therefore, we have "Consistency" verified. In Figure 5.2(b), a labeling of the arcs is given by elements of the group $S_3$ for the same orientation of the trefoil as used in part (a). In Exercise 5.1.9(b) it is to be shown that the labels generate $S_4$ and we have "Generation" satisfied. We can also verify that "Consistency" is also satisfied.



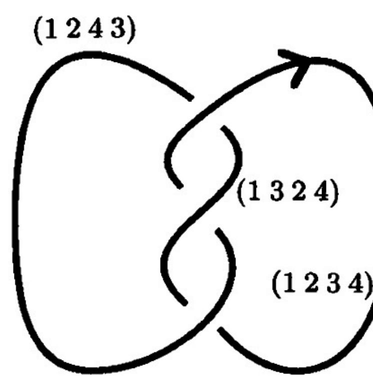




*Figure 5.2*

**Note.** The fact that, in the previous Note, we could label the trefoil knot both with elements of $S_3$ and with elements of $S_4$ is not a coincidence. Kenneth A. Perko proved that a labeling with elements of $S_4$ exists if and only if a labeling with elements of $S_3$ exist, in "Octahedral Knot Covers" in *Knots, Groups, and 3-Manifolds, Papers Dedicated to the Memory of R. H. Fox*, ed. L. P. Neuwirth, pp. 47-50, Princeton, NJ: Princeton University Press (1975) (available on Goggle Books, accessed 2/17/2021).

**Note.** The next result shows that if any oriented diagram of a knot can be labeled with the elements of group $G$, then every oriented diagram of that knot can be labeled with elements of the same group $G$.

**Theorem 5.2.1.** If a diagram for a knot can be labeled with elements from a group $G$, then any diagram of the knot can also be labeled with elements from that group, regardless of the choice of orientation.

**Note.** Livingston declares: "The use of labelings is one of the most powerful means of distinguishing knots." (See page 94.) He describes how M. Thistlethwaite compiled around 12,965 knots with 13 crossings. These knots had 5,639 different Alexander polynomials, by Thistlethwaite used labelings to reduce the list to about 1,000. Livingston also observes the extreme difficulty in finding graph labelings.

*Revised: 2/17/2021*