Section 5.3. Conjugation and the Labeling Theorem

Note. In this section we define conjugate elements in a group and define the cycle structure for elements of the symmetric group S_n. We prove that two elements of S_n are conjugate if and only if they have the same cycle structure (Theorem 5.3.A; the proof of this is not in Livingston).

Note. By Theorem I.6.3 of Thomas W. Hungerford's *Algebra*, Graduate Texts in Mathematics #73, NY: Springer Verlag (1974), every permutation $\sigma \in S_n$ is uniquely (up to the order of the factors) a product of disjoint cycles, each of which has length at least 2 (see my online notes for Modern Algebra 1 on Section I.6. Symmetric, Alternating, and Dihedral Groups; see Theorem I.6.3). This allows us to state the following.

Definition. In a symmetric group S_n, the *cycle structure* of a permutation $\sigma \in S_n$ is the collection of lengths of the cycles in an expression of σ as a product of disjoint cycles as described above.

Definition. Elements g and h in group G are *conjugates* if there is an element $k \in G$ such that $k^{-1}gk = h$. The set of all group elements which are conjugate to element $g \in G$ form the *conjugacy class* containing g.

Note. In the symmetry group S_5, we have that $g = (1,2)(3,4,5)$ is conjugate to $(1,5,3)(2,4)$, since with $k = (1,2,4,5,3)$ (and so $k^{-1} = (1,3,5,4,2)$),

$$k^{-1}gk = (1,3,5,4,2)(1,2)(3,4,5)(1,2,4,5,3) = (1,5,3)(2,4) = h.$$

Theorem 5.3.A. In the symmetric group S_n, two elements are conjugate if and only if they have the same cycle structure.

Note. We can use Theorem 5.3.A to easily find conjugacy classes in symmetry groups. For example, there are seven conjugacy classes in S_5:

1. The conjugacy class of permutations with exactly 3 fixed points; this consists of all transpositions, of which there are $\binom{5}{2} = 10$,

2. The conjugacy class of permutations with exactly 2 fixed points; this consists of all cycles of length 3, of which there are $\binom{5}{2} \times 3!/3 = 20$.

3. The conjugacy class of permutations with exactly 1 fixed point and a cycle of length 4, of which there are $5 \times 4!/4 = 30$.

4. The conjugacy class of permutations with exactly 1 fixed point and 2 transpositions, of which there are $5 \times \binom{4}{2}/2 = 15$.

5. The conjugacy class of permutations with no fixed points and a cycle of length 5, of which there are $5!/5 = 24$.

6. The conjugacy class of permutations with no fixed points, a transposition, and a cycle of length 3, of which there are $\binom{5}{2} \times 3!/3 = 20$.

7. The conjugacy class of 5 fixed points, of which there is only 1 (the identity permutation).

Note. If a knot diagram has been labeled with the elements of a group, then the consistency condition condition implies that the label of the arc that passes under a crossing is conjugate to the one that emerges from the crossing (see Figure 5.1). Since every arc plays the role as an arc passing under a crossing (at one of its ends), then the label of an arc is conjugate to the label of the arc emerging from the crossing and so, as we trace over the knot diagram, all the labels must fall into the same conjugacy class in a group.

Note. The next result shows that if any oriented diagram of a knot can be labeled with the elements of some conjugacy class of a group G, then every oriented diagram of that knot can be labeled with elements of the same conjugacy class of the group G.

Theorem 5.3.2. If a diagram of an oriented knot can be labeled with elements of a group, with the labels coming from some conjugacy class of the group, then every diagram of that knot can be labeled with elements from that conjugacy class.

Note. In Exercise 3.4 it is to be shown that the knot diagram for 9_{46} in Appendix 1 can be labeled with transpositions from S_4, while the diagram for knot 6_1 has no such labeling. Notice that Theorem 5.3.2 then implies that the knots 6_1 and

9_{46} are distinct. This observation is of interest since the Alexander polynomial for both knots is $-2t^2 + 5t - 2$, so that this does not imply that the knots are distinct.

Revised: 2/18/2021