Section 5.4. Equations in Groups and the Group of a Knot

Note. In this section we argue that every knot diagram can be labeled with the elements of a group, as defined in Section 5.2. Knots and Groups. The theoretical justification of our claims are given by the idea of a "free group" and a "group presentation."

Note. Consider the oriented knot diagram in Figure 5.8 (we present an edited version more clearly indicating the orientation).

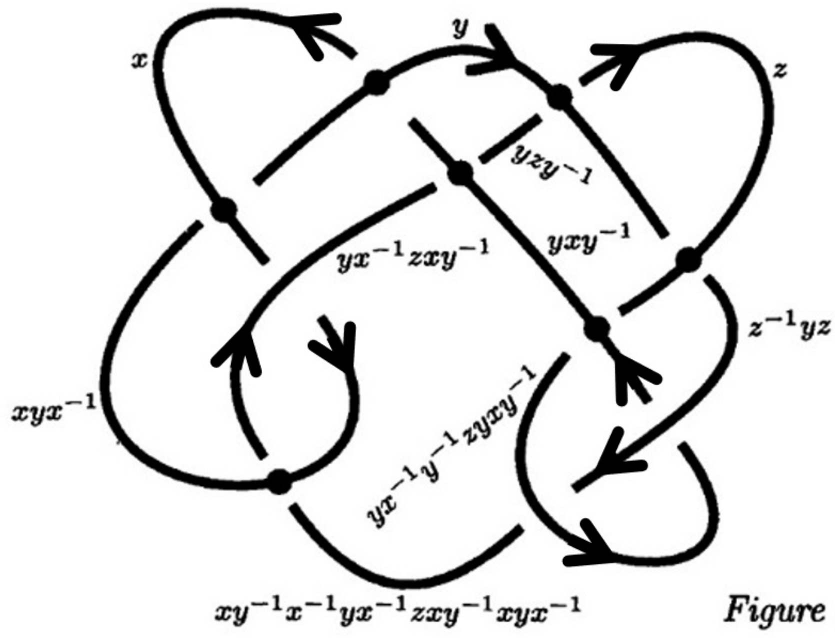


Figure 5.8

Let G be a group and let $x, y, z \in G$. Suppose the arc labels x, y, and z are assigned to the top three arcs, as given.

Example 5.4.A. Let G be a group and let arcs of the oriented graph in Figure 5.8 be labeled $x, y, z \in G$, as given. Then the other given labels on arcs are as presented.

Note. We now check the consistency condition at the three unmarked crossings.

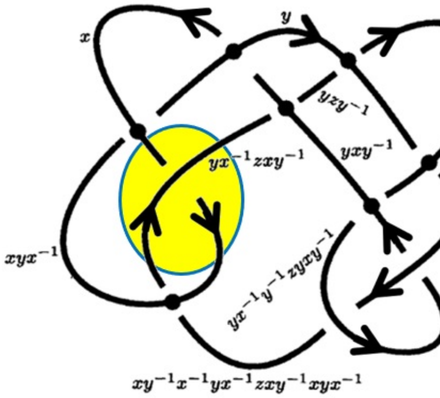
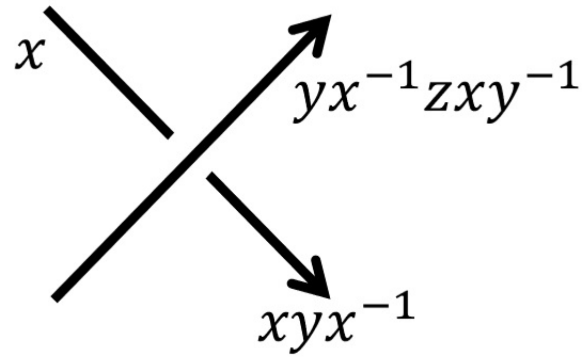


Figure 5.8



The left unmarked crossing is left-handed and

$$(yx^{-1}zxy^{-1})(x)(yx^{-1}zxy^{-1})^{-1} = (xyx^{-1})$$

$$\text{or } (yx^{-1}zxy^{-1})(x)(yx^{-1}z^{-1}xy^{-1}) = xyx^{-1}.$$

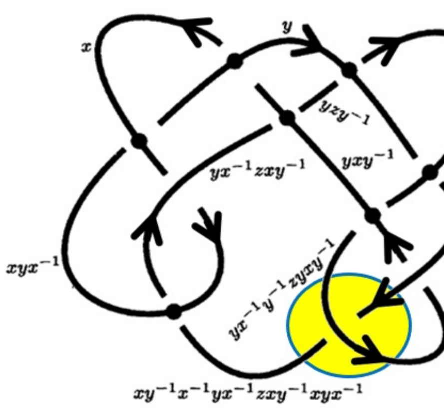
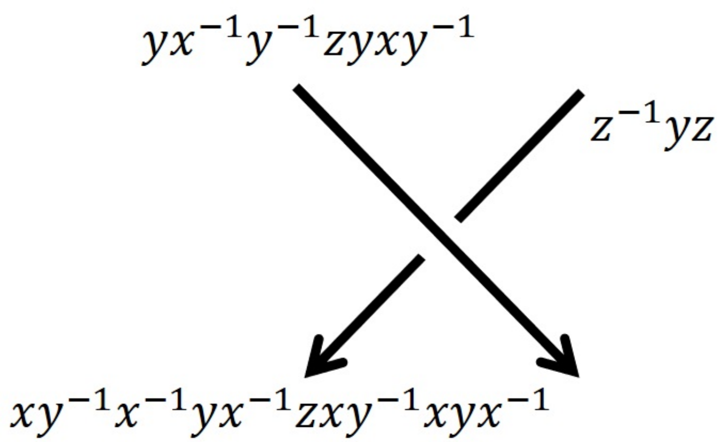


Figure 5.8



The middle unmarked crossing is left-handed and

$$(yx^{-1}y^{-1}zyxy^{-1})(z^{-1}yz)(yx^{-1}y^{-1}zyxy^{-1})^{-1} = (xy^{-1}x^{-1}yx^{-1}zxy^{-1}xyx^{-1})$$

$$\text{or } (yx^{-1}y^{-1}zyxy^{-1})(z^{-1}yz)(yx^{-1}y^{-1}z^{-1}yxy^{-1}) = (xy^{-1}x^{-1}yx^{-1}zxy^{-1}xyx^{-1}).$$

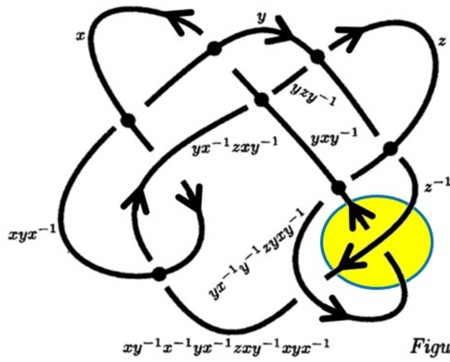
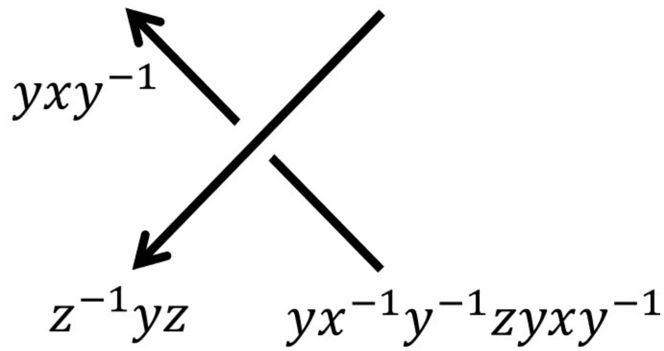


Figure 5.8



The right unmarked crossing is left-handed and

$$(z^{-1}yz)(yx^{-1}y^{-1}zyxy^{-1})(z^{-1}yz)^{-1} = (yxy^{-1})$$

$$\text{or } (z^{-1}yz)(yx^{-1}y^{-1}zyxy^{-1})(z^{-1}y^{-1}z) = (yxy^{-1})$$

(notice that this equation is equivalent to that given by Livingston; he has taken the $(z^{-1}yz)$ and $(z^{-1}yz)^{-1}$ terms to the other side of the equation). Since we are using multiplicative notation group G, then we denote the identity as 1. We can then write the three equations concerning the unmarked crossings as (by multiplying by the inverse of the right-hand side and dropping the parentheses):

$$yx^{-1}zxy^{-1}xyx^{-1}z^{-1}xy^{-1}xy^{-1}x^{-1} = 1, \tag{R1}$$

$$yx^{-1}y^{-1}zyxy^{-1}z^{-1}yzyx^{-1}y^{-1}z^{-1}yxy^{-1}xy^{-1}x^{-1}yx^{-1}z^{-1}xy^{-1}xyx^{-1} = 1, \tag{R2}$$

$$z^{-1}yzyx^{-1}y^{-1}zyxy^{-1}z^{-1}y^{-1}zyx^{-1}y^{-1} = 1 \tag{R3}$$

(the third equation does not look like Livingston's, but the two equations are equivalent as we now show). Notice that relation $R3$ is equivalent to

$$z^{-1}yzyx^{-1}y^{-1}zyxy^{-1}z^{-1}y^{-1}z(yx^{-1}y^{-1}) = 1,$$

or

$$(z^{-1}yz)(yx^{-1}y^{-1}zyxy^{-1})(z^{-1}y^{-1}z) = yxy^{-1},$$

or

$$(yx^{-1}y^{-1}zyxy^{-1}) = (z^{-1}y^{-1}z)(yxy^{-1})(z^{-1}yz)$$

(this is Livingston's third equation), or (moving the left-hand side to the right getting the identity 1 on the left, then interchanging sides)

$$(z^{-1}y^{-1}zyxy^{-1}z^{-1}yz)yx^{-1}y^{-1}z^{-1}yxy^{-1} = 1 \tag{R3$'$}$$

(this is Livingston's third relationship). We have performed this computation of showing that (R3) and (R3′) are equivalent for use in an application in Chapter 6 (see Example 6.4.C).

Note 5.4.A. Finding a graph labeling on a knot diagram with n crossings, is equivalent to solving n equations (those equations implied by the consistency condition) in n variables (the n variables that are the labels of the n arcs in the diagram). However, in the example we just saw we only had three (complicated) equations in three unknowns; of course, along the way we had already solved the equations associated with the marked crossings (or, at least, we solved them in terms of x, y and z). In general, we will start with n variables (one for each arc) and n equations (each resulting from the consistency condition at one of the n crossings). Livingston claims that in general, the procedure results in a set of relations in a collection of relations for which one relation is a consequence of the others. So in solving the equations, one of the relations can be dropped (see page 102). So in the example above, the knot group is determined by only two of the relations (R1), (R2), and (R3) (or is determined by only two of the relations (R1), (R2), (R3′)).

Note. We interpret the labels on the arcs of a knot diagram as symbols and we form "words" of the symbols by taking products of the symbols and their inverses. The symbols are "generators" of a group and the equations that result by setting words equal to 1 (in multiplicative notation) are the "relations" of the group. The generators together with the relations give a "presentation" of a group G. The theoretical backing of this depends on a knowledge of free groups and normal groups. See my online notes for Modern Algebra 1 (MATH 5410) on Section I.9. Free Groups, Free Products, and Generators and Relations; in particular, notice Definition I.9.4 and Notes I.9.1, I.9.2, and I.9.3. A lighter version of this material is also covered in Introduction to Modern Algebra 1 (MATH 4137/5137) in Section VII.39. Free Groups and Section VII.40. Group Presentations. On a personal note, I would like to point out that **there is a "real-world" application of abstract algebra!**

Example 5.4.A. Exercise I.9.8 of Thomas W. Hungerford's *Algebra* (Graduate Texts in Mathematics #73, NY: Springer-Verlag (1974)) claims that $\langle X \mid Y \rangle$ where $X = \{a, b\}$ and $Y = \{a^n, b^2, abab\}$ (for $n \geq 3$) is a presentation of the dihedral group D_n of symmetries of a regular n-gon. Element a represents an elementary rotation of the n-gon (so it is of order n and $a^n = 1$ as required) and element b is a "flip" of the n-gon about and axis (so it is of order 2 and $b^2 = 1$ as required. The word $abab$ yields the relation $abab = 1$ which describes how the rotations and flips interact.

Definition. Given a knot diagram, the group that results from the generators (arc labels) and the words (more accurately, the relations) determined by the consistency condition is called a *group of the knot*.

Note. The knot with diagram given in Figure 5.8 has group G where a presentation of G, using relations (R1), (R2), and (R3), is given by

$$G = \langle x, y, z \mid yx^{-1}zxy^{-1}xyx^{-1}z^{-1}xy^{-1}xy^{-1}x^{-1},$$

$$yx^{-1}y^{-1}zyxy^{-1}z^{-1}yzyx^{-1}y^{-1}z^{-1}yxy^{-1}xy^{-1}x^{-1}yx^{-1}z^{-1}xy^{-1}xyx^{-1},$$

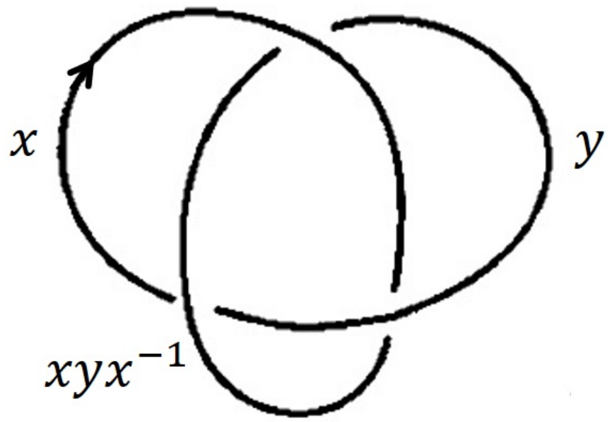
$$z^{-1}yzyx^{-1}y^{-1}zyxy^{-1}z^{-1}y^{-1}zyx^{-1}y^{-1}\rangle.$$

In light of Note 5.4.A, we can simplify the presentation of group G and use only (R1) and (R3):

$$G = \langle x, y, z \mid yx^{-1}zxy^{-1}xyx^{-1}z^{-1}xy^{-1}xy^{-1}x^{-1},$$

$$z^{-1}yzyx^{-1}y^{-1}zyxy^{-1}z^{-1}y^{-1}zyx^{-1}y^{-1}\rangle.$$

Example 5.4.B. Consider the trefoil knot with the orientation given below. The upper most crossing is a right-handed crossing and so we get that xyx^{-1} must be the label of the third arc, as given in the figure. Next, the lower left crossing is a right-handed crossing and so we need $(xyx^{-1})x(xyx^{-1})^{-1} = y$ or $xyx^{-1}xxy^{-1}x^{-1} = y$ or $xyxx^{-1}x^{-1} = y$ or $xyxy^{-1}x^{-1}y^{-1} = 1$. The lower right crossing is a right-handed so we need $y(xyx^{-1})y^{-1} = x$ or $yxyx^{-1}x^{-1} = 1$ or $(yxyx^{-1}x^{-1})^{-1} = (1)^{-1}$ or $xyxy^{-1}x^{-1}y^{-1} = 1$ (as we have already seen). So the trefoil has group G where a presentation of G is given by $\langle x, y \mid xyxy^{-1}x^{-1}y^{-1}\rangle$.



Note. Livingston makes several claims on pages 103 and 104. He claims:

1. Although the group of a knot depends on the choice of diagram and the choice of generators, "it can be proved" that any two groups of a knot are isomorphic.

2. If a knot group is isomorphic to $\mathbb{Z}$ (the additive group of the integers), then the knot is trivial.

3. Nonequivalent prime knots have nonisomorphic knot groups.

The first claim show that the group of a knot is an invariant. That is, if two knots have nonisomorphic groups then the knots are not equivalent (though nonequivalent knots can have the same group; this does not hold for prime knots, as state in the third claim). The second claim is related to the Dehn Lemma (an error in the proof of which was found by Kneser and a correct proof was given by Papakyriakopoulos; the history of this is described in Chapter 1. A Century of Knot Theory in the notes for Livingston's book). A proof is given in Corollary 11.3 of W. B. Raymond Lickorish, *An Introduction to Knot Theory*, Graduate Texts in

Mathematics #175, NY: Springer (1997) (though the notation looks different; we will explain this difference in the next section).

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