Section 5.5. The Fundamental Group

Note. In the previous section, we associated a group with a graph by labeling the arcs of a graph diagram. In this section, we associate a "fundamental group" with a knot K by considering the space $\mathbb{R}^3 - K$ (that is, the set of points in 3-space that are *not* on knot K). This group is based on the homotopy of paths. Path homotopy is discussed in my online notes for Algebraic Topology (not a formal ETSU class, but one that might follow Introduction to Topology [MATH 4357/5357]) in Section 51. Homotopy of Paths. This is also addressed in Complex Analysis 1 (MATH 5510) in IV.6. The Homotopic Version of Cauchys Theorem and Simple Connectivity. The fundamental group is discussed in general in Section 52. The Fundamental Group, and for surfaces in particular in Section 60. Fundamental Groups of Some Surfaces.

Note. The sources above give rigorous definitions of the topics at hand. In these notes, we give informal "definitions" and rely on intuition and pictures.

"Definition." For K a knot in 3-space, let $X = \mathbb{R} - K$. Pick a point p in X (caled a *base point*). Consider the set of all *closed oriented paths* in X that begin and end at p. Two such paths are *homotopic* if one can be continuously transformed to the other in X while keeping the ends at point p throughout the continuous transformation. A set containing all such paths which are homotopic to each other form a *homotopy class*. The *fundamental group* of knot K is the group formed by the homotopy classes of paths that begin and end at p, where the binary operation is the concatenation of homotopy classes (that is, following one path by another).

Note. A number of concerns arise immediately. Without giving details, we claim that path homotopy is an equivalence relation so that a homotopy class is an equivalence class of the equivalence relation of homotopy. The binary operation is well-defined by representatives of homotopy classes and the homotopy classes do in fact form a group under this binary operation. The identity homotopy class contains contains the constant path p, and the inverse of path $\gamma = \gamma(t)$ (where t is a parameter in a parameterization of t) is $\gamma^{-1} = \gamma(-t)$.

Note. We have not required the paths to be simple and they may intersect themselves. Since paths can self intersect, then in a homotopy a path may intersect itself, unlike the deformation of a knot. Since the homotopy is a mapping within $X = \mathbb{R}^3 - K$, then in a homotopy the path cannot cross knot K.

Note. A knot diagram is given in Figure 5.10, along with three paths γ_1, γ_2, and γ_3 which begin and end at given point p. Path γ_1 and γ_2 are homotopic, but path γ_3 is homotopic to neither γ_1 nor γ_2. Here, we speak informally and do not give rigorous proofs of these claims.

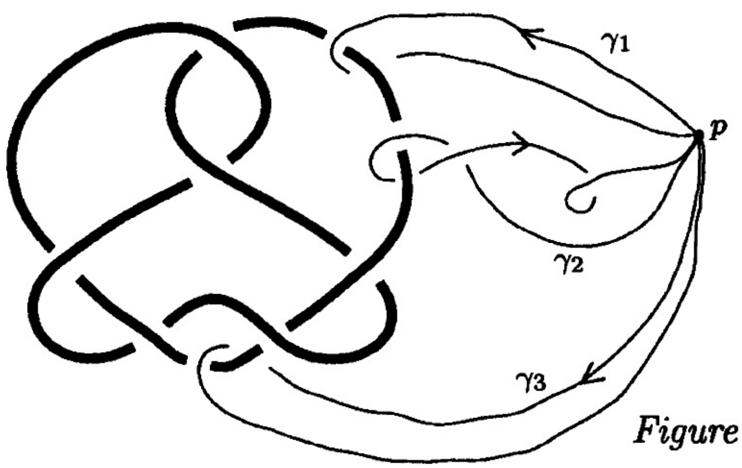

Figure 5.10

Note. In Figure 5.11, we see how the binary operation in the fundamental group is performed on γ_1 and γ_3 (strictly speaking, the binary operation is on the homotopy classes containing these representatives and Figure 5.11 gives a representative of the product). We need specific parameterizations of γ_1 and γ_3 in order to calculate a parameterization of the product $\gamma_1\gamma_3$; in so doing we would see that $\gamma_1\gamma_3$ first traces out γ_1 first (say) and then traces out γ_3, whereas $\gamma_3\gamma_1$ races out γ_3 first and then traces out γ_1 so that the fundamental group is not commutative (not "abelian").

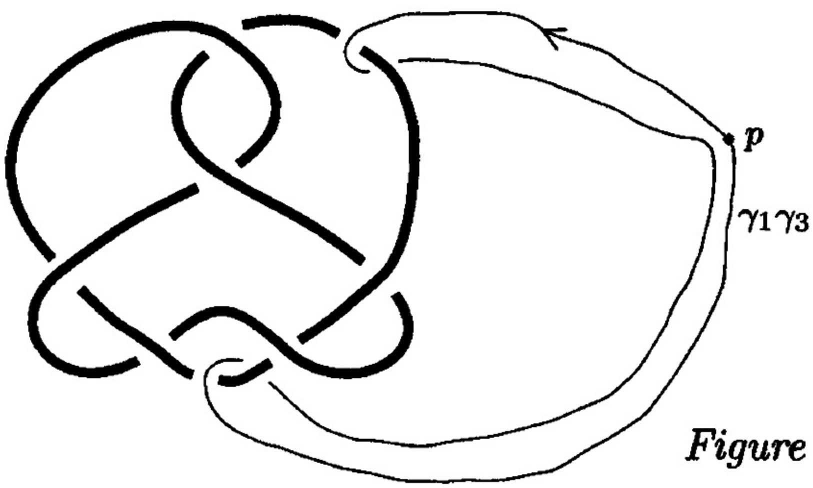

Figure 5.11

Note. Livingston makes three comments at the end of this section of the nature "it can be proved..." (see pages 107 and 108).

Note 5.5.A. The Van Kampen Theorem (also called the Seifert-Van Kampen Theorem): "expresses the structure of the fundamental group of a topological space X in terms of the fundamental groups of two open, path-connected subspaces that cover X. It can therefore be used for computations of the fundamental group of spaces that are constructed out of simpler ones." This quote if from the Wikipedia page on the Seifert-Van Kampen Theorem; see also "Section 70. The Seifert-van Kampen Theorem" in James R. Munkres' *Topology*, 2nd Edition, Upper Saddle River, NJ: Prentice Hall (2000). Livingston states: "A diagram of a knot yields a decomposition of the knot complement which, using the Van Kampen Theorem, in turn produces a simple presentation of the fundamental group. That presentation is the same as the presentation of the knot group described in the previous section." Therefore, the group of a knot as defined in the previous section based on labeling the arcs of a knot diagram, is the same of the fundamental group! So we are justified in referring to *the* group of a knot.

Note 5.5.B. In a knot diagram, for each arc there is an element of the fundamental group which is represented by a path from point p, directly to the arc, once around the arc, and then back to point p (there is plenty of room in 3-space, so such paths exist). Livingston states: "That element corresponds to the element in the knot group given by the variable label on the arc. It can be proved that relations between the elements in the fundamental group correspond to the relations in the knot group arising at the crossings." So, again, we see that the group of a knot based on a labeling is the same as the fundamental group.

Note 5.5.C. Livingston also relates homomorphisms of the fundamental group with the labeling of the arcs of a knot diagram as follows: "Given a homomorphism of the fundamental group of a knot complement, composing it with the correspondence between the knot group and the fundamental group gives an assignment of an element in G to each arc in the diagram. That is, labelings of the diagram turn out to correspond to homomorphisms of the fundamental group of the knot complement. The consistency condition on the labeling corresponds to the map being a homomorphism. The generation condition corresponds to the map being surjective [onto]."

Revised: 2/20/2021