Chapter 6. Geometry, Algebra, and the Alexander Polynomial

Note. In Section 3.5 we defined the Alexander polynomial and the Alexander matrix of a knot. In this chapter we give algebraic and geometric approaches to Alexander polynomial. The geometric approach involves the Seifert matrix, to be introduced in Section 6.1.

Section 6.1. The Seifert Matrix

Note. In this section, we define the linking number of an oriented link with two components. We consider closed oriented paths on a Seifert surface and use the idea of the linking number to define a matrix, called the Seifert matrix. We illustrate it with an example. Most of the definitions of this section are rather informal.

Note. In Figure 6.1 we have a disk to which has been added four bands. Notice that oriented closed curves appear on the surface which run from the disk, along the "core" of a band, and back to the disk. If the surface is a Seifert surface of a knot, then the twisting and linking of these curves contain information about the knot. We encode such information in the Seifert matrix below.

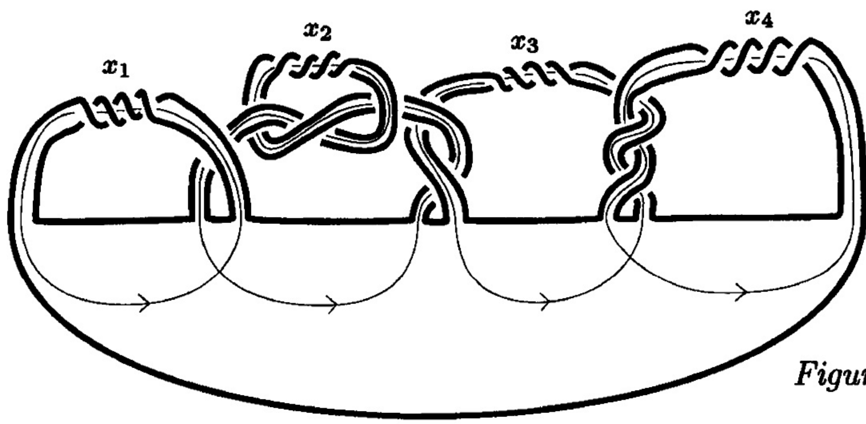

Figure 6.1

Definition. Consider a diagram of an oriented link with two components, K and J. With each crossing in the diagram, associate the number $+1$ is the crossing is right-handed and associate the number -1 if the crossing is left-handed (where left-handed and right-handed crossing are as described in Figure 5.1 of Section 5.2; see also Figure 3.7 in Exercise 3.2.5). These values are the *signs* of the crossing. The *linking number* of K and J, denoted $\ell k(K, J)$, is the sum of the signs of the crossing points where K and J meet, divided by 2.

Note. In Exercise 3.2.5 (where the definition of linking number first appears), it is to be shown that the the linking number depends only on the oriented knot and not on the diagram used to compute it. That is, the linking number is an (oriented knot) invariant. Notice that the roles of K and J are interchangeable, so that $\ell(K, J) = \ell(J, K)$.

Note. For a given knot K and Seifert surface F of the knot, we know by definition that F is an orientable surface. So, informally, it has "one side" which we can label the "top" of the surface. More formally, this is dealt with in terms of nonvanishing normal vectors to the surface (which always point "up").

"Definition." Let K be a given knot K and let F be a Seifert surface of K. Choose a side of surface F as the top side. For any given simple oriented closed curve x on F, a *positive push off* of x, denoted x^*, which runs "parallel" to x and lies just "above" F.

Note. We need the idea of a positive push off so that we can consider crossings between two paths on the Seifert surface. So we can compute the linking number between a oriented closed curve x on F and the positive push off y^* of another oriented closed curve on F. We need the curves on F to intersect only at isolated points of F and when two curves intersect we need them to cross (as opposed to being tangent).

Note 6.1.A. By Theorem 4.2.3, if a connected orientable surface is formed by attaching bands to a collection of disks then the genus of the resulting surface is

$$(2 - \#\text{disks} + \#\text{bands} - \#\text{boundary components})/2.$$

So if F is a Seifert surface of a knot, then it has only one boundary component and if it is formed by adding bands to a single disk, then the genus is $(2 - (1) + \#\text{bands} - (1))/2 = \#\text{bands}/2$, so that the number of bands is twice the genus.

Since each band will have a simple oriented closed curve associated with it, the the number of curves is twice the number of bands. For example, in Figure 6.1 we started with a single disk and added 4 bands, so that the genus of the surface is 2 and the number of simple oriented closed curves is 4.

Definition. Let F be a Seifert surface of a knot K that is formed from a single disk by adding bands. If F is genus g then $2g$ bands are present (as described in Note 6.1.A above) and so there are $2g$ simple oriented closed curves $x_1, x_2, \ldots, x_{2g}$ associated with surface F. The *Seifert matrix* V of surface F is the $2g \times 2g$ matrix with (i,j)-entry of $v_{i,j} = \ell k(x_i, x_j^*)$.

Note. The Seifert matrix depends on a number of choices (such as the orientations of the curves and the choice of the "top" side of F) and, as Livingston claims on page 112, "by itself is not an invariant of the knot." In Sections 6.2 and 6.3 we will use the Seifert matrix to define knot invariants, including the Alexander polynomial.

Example. We now consider the Seifert matrix of the Seifert surface given in Figure 6.1 above. We interpret the top of the surface to be the part contained in the disk which faces us in Figure 6.1. In Figure 6.2, we have the curves x_2 and x_3^*. Notice that the "twisting" of the band which contains x_2 is lost in path x_2 (since x_2 is at the "core" or center of the band, say), but the knottedness of this band is reflected in Figure 6.2, and similarly for the band containing x_3; the crossings generated by

x_3^* and are also reflected in Figure 6.2.

Figure 6.2

Both crossings are right-handed, so the linking number is $\ell k(x_2, x_3^*) = ((+1) + (+1))/2 = 1$. So in the Seifert matrix, $v_{2,3} = 1$. Figure 6.3 gives the curves x_2 and x_2^* (the book's version is modified here to give x_2^* as the blue curve). Notice that the twisting of the band containing path x_2 has an influence this time because it causes x_2^* to wrap around x_2 twice (you might follow one of the boundaries of the band to convince yourself of this). The knottedness of the band is reflected here, similar to the case given in Figure 6.2.

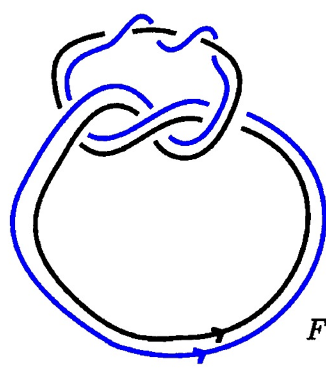

Figure 6.3

There are 10 crossings of x_2 and x_2^* (follow the blue curve and see how many times it crosses the black curve), and all 10 are left-handed. So the linking number is $\ell k(x_2, x_2^*) = (10 \times (-1))/2 = -5$. So in the Seifert matrix, $v_{2,2} = -5$. As described in Note 6.1.A, the genus of the Seifert surface is 2, so that the Seifert matrix is

4×4. The other entries of the matrix are to be found in Exercise 6.1.2. We find that the Seifert matrix for the surface of Figure 6.1 is

$$V = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & -2 \end{pmatrix}.$$

(Notice that the $(2,1)$-entry of V as given in the book is incorrect. When we consider x_2 and x_1^* there are two crossings; one is left-handed and one is right-handed. So $v_{2,1} = \ell k(x_2, x_1^*) = 0$.)

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