Section 6.2. Seifert Matrices and the Alexander Polynomial

Note. In this section, we state (without proof) a theorem that relates the Alexander polynomial to the Seifert matrix.

Note. Livingston declares a proof of the next result to be "not at all evident" (see page 115). It seems that in some sources, this is the definition of the Alexander polynomial; see the bottom of page 55 of W. B. Raymond Lickorish's *An Introduction to Knot Theory*, Graduate Texts in Mathematics #175, NY: Springer (1997). When we refer to "the" Alexander polynomial, remember that we need a standard version of an Alexander polynomial, as explained in Note 3.5.B.

Theorem 6.2.1. Let V be a Seifert matrix for a knot K, and V^t is transpose. The Alexander polynomial is given by the determinant $\det(V - tV^t)$. If K has a genus g Seifert surface, then V is $2g \times 2g$ (by definition), so an upper bound on the degree of the Alexander polynomial is $2g$.

Note. Using the properties of determinants, we can easily prove the following.

Corollary 6.2.2. The Alexander polynomial of a knot K satisfies $A_K(t) = t^{\pm i} A_K(t^{-1})$ for some $i \in \mathbb{Z}$.

Example. In the previous section we saw that the Seifert matrix of the knot of Figure 6.1 is (remember that there is an error in the book's version of this matrix):

$$V = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & -2 \end{pmatrix}.$$

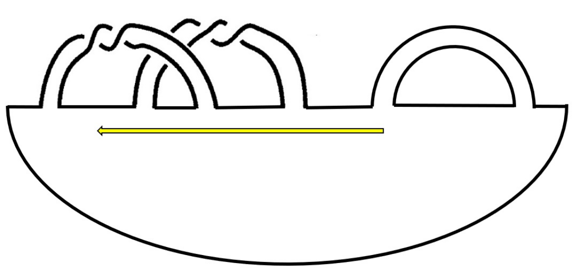
So we have by Theorem 6.2.1, the Alexander polynomial of this knot is

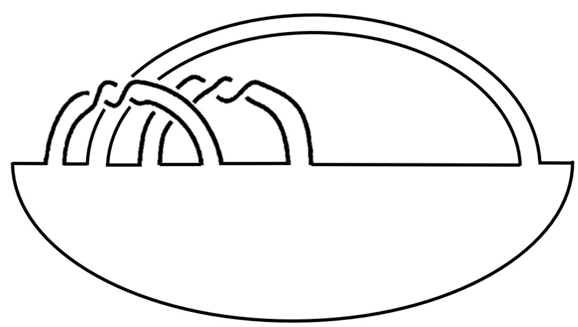
$$\det(V - tV^t) = \det \left(\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & -2 \end{pmatrix} - t \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & -5 & 1 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -1 & -2 \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} 2-2t & 1 & 0 & 0 \\ -t & -5+5t & 1-t & 0 \\ 0 & 1-t & 2-2t & -1+2t \\ 0 & 0 & -2+t & -2+2t \end{pmatrix} = 64t^2 - 272t^3 + 217t^2 - 272t + 64.$$

Definition. For a Seifert Surface which is a disk with bands added, a *band move* results from sliding one of the points at which a band is attached over another band.

Note 6.2.A. The following figure gives an example of a band move. A band move results in another surface that is a disk with bands attached. The $2g$ curves formed from the cores of the new bands will be different from the cores of the original bands and will interact in a different way, hence changing the Seifert matrix. Livingston claims without proof that the effect of a band move on the Seifert matrix V is, for some i and j, to add a multiple of the ith row to the j row and then to add the same multiple of the ith column to the jth column. Now each elementary row operation can be performed by multiplying on the left by an appropriate elementary matrix (see my online notes for Linear Algebra [MATH 2010] on 1.4. Solving Systems of Linear Equations; see Theorem 1.8). Also, each elementary column operation can be performed by multiplying on the right be an appropriate elementary matrix (see my online notes for Theory of Matrices [MATH 5090] on Section 3.2. Multiplication of Matrices and Multiplication of Vectors and Matrices; see Theorem 3.2.3). Since the same same operation is applied to the rows as the columns, then a sequence of band moves results in changing the Seifert matrix V to the matrix MVM^t where M is some invertible matrix with integer entries (M and M^t are products of the elementary matrices representing the sequence of elementary row and column operations).





Definition. For a Seifert Surface which is a disk with bands added, when two bands are added to the disk such that (1) one band is untwisted and unknotted, and (2) the other band may be twisted or knotted and can link with the other bands, then this procedure is called *stabilization*.

Note. Figure 6.7 illustrates the stabilization procedure. Livingston claims that the boundary of the new surface after stabilization is "clearly" the same knot as the original Seifert surface, and that the Seifert matrix of the new surface results from the original Seifert matrix V with two new columns and rows added as follows:

$$\begin{pmatrix} & & & * & 0 \\ & V & & \vdots & \vdots \\ & & & * & 0 \\ * & \cdots & * & * & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

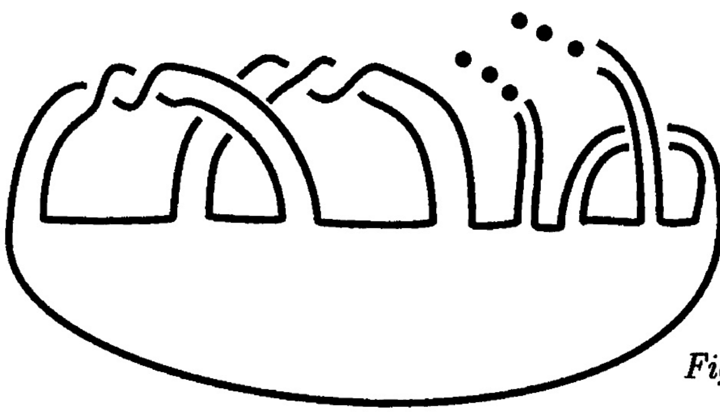


Figure 6.7

Definition. Two matrices with integer entries are *S-equivalent* if they differ by a sequence of the two operations given above associated the Seifert matrix corresponding to bond moves and stabilization (or their inverses).

Note. Livingston claims that a "difficult geometric argument" shows that for any two Seifert surfaces for a knot, there is a sequence of stabilizations that be be applied to each so that the resulting surfaces can be deformed into each other. The next result summarizes this in terms of the Seifert matrices.

Theorem 6.2.3. Any two Seifert matrices for a knot are S-equivalent.

Note. The next result is easy to prove using properties of determinants.

Corollary 6.2.4. If V_1 and V_2 are Seifert matrices associated with the same knot, then the polynomials $\det(V_1 - tV_1^t)$ and $\det(V_2 - tV_2^t)$ differ by a multiple of $\pm t^k$.

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