Section 6.3. Signature of a Knot, and Other S-equivalent Invariants

Note. In this section, we give some knot invariants that are determined by a Seifert matrix of a knot.

Note 6.3.A. In the proof of Corollary 6.2.4 we saw that if V' results from V by a single step of stabilization, the $\det(V' - t(V')^t) = t\det(V - tV^t)$. So for V a Seifert matrix of a knot, the determinant of the symmetric matrix $V + V^t$ is changed by a negative sign by stabilization (take $t = -1$ in the above equality). Now a band move has the effect of multiplying V by an elementary matrix M of determinant 1 to give MVM^t. Since $\det(MVM^t) = \det(M)\det(V)\det(M^t) = \det(M)\det(V)\det(M) = (1)(\det(V))(1) = \det(V)$, the second manipulation of the Seifert matrix does not change the determinant. So if two matrices V_1 and V_2 are S-equivalent then the determinants of $V_1 + V_1^t$ and $V_2 + V_2^t$ differ at most by a multiple of -1. So for V a Seifert matrix of a knot, the quantity $|\det(V + V^t)|$ is a knot invariant. Livingston claims that this invariant is the same as the determinant of a knot defined in Section 3.4 and that it is the absolute value of the Alexander polynomial evaluated at $t = -1$ (see page 119).

Definition. Every real symmetric matrix A is diagonalizable; the diagonalization $A = CDC^{-1}$ can be achieved by using a real orthogonal matrix C (see my online notes for Linear Algebra [MATH 2010] on 6.3 Orthogonal Matrices; see Theorem 6.8. Fundamental Theorem of Real Symmetric Matrices). The *signature* of matrix

A is the number of positive entries minus the number of negative entries on the diagonal. For a Seifert matrix V of a knot K, the matrix $V+V^t$ is symmetric and its signature is the *signature* of knot K, denoted $\sigma(K)$.

Note. W. B. Raymond Lickorish, in *An Introduction Knot Theory*, Graduate Texts in Mathematics 175, (NY: Springer-Verlag, 1997) "Chapter 8 The Conway Polynomial, Signatures and Slice Knots," give a definition of the signature of a knot and give the signature of all knots with less than eight crossings (see Table 8.1 on his page 85).

Example. We saw in Section 6.1 that a Seifert matrix for the graph of Figure 6.1 is

$$V = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

(with the $(2,1)$-entry corrected, as explained in the notes for Section 6.1). So the matrix

$$A_1 = V + V^t = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & -10 & 2 & 0 \\ 0 & 2 & 4 & -3 \\ 0 & 0 & -3 & -4 \end{pmatrix}$$

is symmetric. Livingston considers this matrix in this section, but unfortunately drops a negative sign in the $(2,2)$-entry (and then does computation based on the incorrect entry). We can add $-1/4$ times the first row to the second row and then add $-1/4$ times the first column to the second column to get

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & -10 & 2 & 0 \\ 0 & 2 & 4 & -3 \\ 0 & 0 & -3 & -4 \end{pmatrix} \sim \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -41/4 & 0 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & -3 & -4 \end{pmatrix} = B.$$

In Exercise 6.3.1, it is to be shown that continuing this process we find that

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & -10 & 2 & 0 \\ 0 & 2 & 4 & -3 \\ 0 & 0 & -3 & -4 \end{pmatrix} \sim \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -41/4 & 0 & 0 \\ 0 & 0 & 180/41 & 0 \\ 0 & 0 & 0 & -121/20 \end{pmatrix} = B,$$

where the row equivalence is accomplished by a sequence row additions and corresponding column additions (so that $B = MAM^t$ where $\det(M) = 1$). Again, notice the discrepancy with Livingston (see his page 120). Since there are 2 positive entries and 2 negative entries on the diagonal, the signature of this matrix is $2 - 2 = 0$. Therefore, for knot K in Figure 6.1 we have $\sigma(K) = 0$ (since the signature of A and B are the same, as shown in the next note).

Note 6.3.B. "Sylvester's Law of Inertia" is named for James J. Sylvester, September 3, 1814–March 15, 1897, who published the result in "A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares" *Philosophical Magazine*, 4th Series. **4**(23), 138142 (1852); an online copy is available at Andrew Ranicki webpage (accessed 2/27/2021). The result implies that if

B is a symmetric matrix given by $B = MAM^t$, where M is invertible, then the signatures of A and B are equal.

Theorem 6.3.5. For a knot K, the value of $\sigma(K)$ does not depend on the choice of Seifert matrix, and is hence a well-define knot invariant.

Example 6.3.A. In Exercise 6.1.1, it was to be shown that a Seifert matrix of the right-handed trefoil knot is $V = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. So we have $V + V^t = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$. By adding $-1/2$ times the first row to the second row and then $-1/2$ times the first column to the second column we get

$$V + V^t = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \sim \begin{pmatrix} -2 & 0 \\ 0 & -3/2 \end{pmatrix}.$$

So the signature of the right-handed trefoil knot is -2. We can similarly show that the left-handed trefoil knot has signature $+2$, so that by Theorem 6.3.5, the left-handed and right-handed trefoil knots are not equivalent.

Note. Recall that a square matrix H with complex entries is Hermitian if it equals its conjugate transpose (denoted H^*), $H = H^*$. See my online notes for Linear Algebra (MATH 2010) on 9.2. Matrices and Vector Spaces with Complex Scalars (see Definition 9.4). Any Hermitian matrix can be diagonalized by performing a sequence of row and column operations. The difference in the complex setting from the real setting is that the of a row operation involving multiplication be complex number z is followed by a column operation related to multiplying the column by the complex conjugate $\overline{z}$. Once diagnonalized, the matrix will have real entries (it diagonal matrix will also be Hermitian and the diagonal entries of all Hermitian matrices must be real). The signature can then be computed as in the real case. For some theoretical support for these claims, see my online Linear Algebra notes on 9.3. Eigenvalues and Diagonalization; in particular, see "Theorem 9.5. The Spectral Theorem for Hermitian Matrices." These notes are part of the study of linear algebra, but are not usually covered in the sophomore class because of time constraints.

Note. As with Sylvester's Law of Inertia for real symmetric matrices, a similar result holds for complex Hermitian matrices. This allows us to use complex numbers to define an additional type of signature for a knot, as follows.

Definition. Let V be the Seifert matrix for a knot K and let ω be a complex number of modulus 1, $|\omega| = 1$. Consider the Hermitian matrix $(1-\omega)V + (1-\omega^{-1})V^t$. The signature of this matrix is the *ω-signature* of K. Interpreting $\omega \in \{z \mid z \in \mathbb{C}, |z| = 1\}$ as a variable, for a given knot K we can define the *signature function* of K on the unit circle in $\mathbb{C}$; we have the mapping $\omega \mapsto \omega$-signature of K.

Revised: 5/4/2021