Section 6.4. Knot Groups and the Alexander Polynomial

Note. In this section, we present an algorithm that computes the Alexander polynomial of a knot based on the group presentation of the knot group as discussed in Section 5.4. Equations in Groups and the Group of a Knot.

Note. The algorithm we present was developed by Ralph Fox in five papers in the 1950s:

1. Fox, R., "Free Differential Calculus, I: Derivation in the Free Group Ring," *Annals of Mathematics*, **57**(3), 547-560 (1953).

2. Fox, R., "Free Differential Calculus, II: The Isomorphism Problem of Groups," *Annals of Mathematics*, **59**(2), 196-210 (1954).

3. Fox, R., "Free Differential Calculus, III: Subgroups," *Annals of Mathematics*, **64**(2), 407-419 (1956).

4. Chen, K-T, R. Fox, and R. Lyndon, "Free Differential Calculus, IV: The Quotient Groups of the Lower Central Series," *Annals of Mathematics* **68**(1), 81-95 (1958).

5. Fox, R., "Free Differential Calculus, V: The Alexander Matrices Re-Examined," *Annals of Mathematics*, **71**(3), 408-422 (1960).

"Free differential calculus" is today known as "Fox calculus." Another source on this by Fox is in Richard Crowell and Ralph Fox, *Introduction to Knot Theory*, NY: Springer Verlag (1963); see "Chapter VII. The Free Calculus and the Elementary Ideals" (pages 94–109).

Definition. Let $x_1, x_2, \ldots, x_n$ be noncommuting variables. A *word* in these variables is a monomial in these variables (i.e., a product of powers of the variables). The *Fox derivative*, $\partial/\partial x_i$, on these words satisfies the rules:

Rule 1. $\dfrac{\partial}{\partial x_i}[x_i] = 1$, $\dfrac{\partial}{\partial x_i}[x_j] = 0$ for $i \neq j$, and $\dfrac{\partial}{\partial x_i}[1] = 0$.

Rule 2. $\dfrac{\partial}{\partial x_i}[wz] = \dfrac{\partial}{\partial x_i}[w] + w\dfrac{\partial}{\partial x_i}[z]$, where w and z are words in variables x_j and x_j^{-1}.

Note. By Rule 1, $\dfrac{\partial}{\partial x_i}[x_i x_i^{-1}] = \dfrac{\partial}{\partial x_i}[1] = 0$, and by Rule 2,

$$\frac{\partial}{\partial x_i}[x_i x_i^{-1}] = \frac{\partial}{\partial x_i}[x_1] + x_i\frac{\partial}{\partial x_i}[x_i^{-1}] = 1 + x_i\frac{\partial}{\partial x_i}[x_i^{-1}].$$

Therefore, $\dfrac{\partial}{\partial x_i}[x_i^{-1}] = -x^{-1}$.

Example 6.4.A. Let $x_1 = x$ and $x_2 = y$. Consider the word $xyxy^{-1}x^{-1}y^{-1}$. We can repeatedly apply Rule 2 to differentiate the word with respect to x and with respect to y. Differentiating with respect to x we have

$$\frac{\partial}{\partial x}[xyxy^{-1}x^{-1}y^{-1}] = \frac{\partial}{\partial x}[(x)(yxy^{-1}x^{-1}y^{-1})] = \frac{\partial}{\partial x}[x] + x\frac{\partial}{\partial x}[yxy^{-1}x^{-1}y^{-1}]$$

$$= 1 + x\frac{\partial}{\partial x}[(y)(xy^{-1}x^{-1}y^{-1})] = 1 + x\left(\frac{\partial}{\partial x}[y] + y\frac{\partial}{\partial x}[xy^{-1}x^{-1}y^{-1}]\right)$$

$$= 1 + x(0) + xy\frac{\partial}{\partial x}[(x)(y^{-1}x^{-1}y^{-1})] = 1 + xy\left(\frac{\partial}{\partial x}[x] + x\frac{\partial}{\partial x}[y^{-1}x^{-1}y^{-1}]\right)$$

$$= 1 + xy(1) + xyx\frac{\partial}{\partial x}[(y^{-1})(x^{-1}y^{-1})] = 1 + xy + xyx\left(\frac{\partial}{\partial x}[y^{-1}] + y^{-1}\frac{\partial}{\partial x}[x^{-1}y^{-1}]\right)$$

$$= 1 + xy + xyx(0) + xyxy^{-1}\frac{\partial}{\partial x}[x^{-1}y^{-1}] = 1 + xy + xyxy^{-1}\frac{\partial}{\partial x}[x^{-1}y^{-1}]$$

$$= 1 + xy + xyxy^{-1}\left(\frac{\partial}{\partial x}[x^{-1}] + x^{-1}\frac{\partial}{\partial x}[y^{-1}]\right) = 1 + xy + xyxy^{-1}(-x^{-1} + x^{-1}(0))$$
$$= 1 + xy - xyxy^{-1}x^{-1}.$$

Differentiating with respect to y we have

$$\frac{\partial}{\partial y}[xyxy^{-1}x^{-1}y^{-1}] = \frac{\partial}{\partial y}[(x)(yxy^{-1}x^{-1}y^{-1})] = \frac{\partial}{\partial y}[x] + x\frac{\partial}{\partial y}[yxy^{-1}x^{-1}y^{-1}]$$
$$= 0 + x\frac{\partial}{\partial y}[(y)(xy^{-1}x^{-1}y^{-1})] = x\left(\frac{\partial}{\partial y}[y] + y\frac{\partial}{\partial y}[xy^{-1}x^{-1}y^{-1}]\right)$$
$$= x(1) + xy\frac{\partial}{\partial y}[(x)(y^{-1}x^{-1}y^{-1})] = x + xy\left(\frac{\partial}{\partial y}[x] + x\frac{\partial}{\partial y}[y^{-1}x^{-1}y^{-1}]\right)$$
$$= x + xy(0) + xyx\frac{\partial}{\partial y}[(y^{-1})(x^{-1}y^{-1})] = x + xyx\left(\frac{\partial}{\partial y}[y^{-1}] + y^{-1}\frac{\partial}{\partial y}[x^{-1}y^{-1}]\right)$$
$$= x + xyx(-y^{-1}) + xyxy^{-1}\frac{\partial}{\partial y}[x^{-1}y^{-1}] = x - xyxy^{-1} + xyxy^{-1}\left(\frac{\partial}{\partial y}[x^{-1}] + x^{-1}\frac{\partial}{\partial y}[y^{-1}]\right)$$
$$= x - xyxy^{-1} + xyxy^{-1}((0) + x^{-1}(-y^{-1})) = x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1}.$$

Note. In Example 5.4.B we saw that a presentation of the group of the trefoil knot is $\langle x, y \mid xyxy^{-1}x^{-1}y^{-1}\rangle$. In Example 3.5.1 we saw that the Alexander polynomial of the trefoil knot is $t^2 - t + 1$. Notice that if we $x = y = t$ in

$$\frac{\partial}{\partial x}[xyxy^{-1}x^{-1}y^{-1}] = 1 + xy - xyxy^{-1}x^{-1}$$

then we get

$$(1 + xy - xyxy^{-1}x^{-1})|_{x=y=t} = 1 + (t)(t) - (t)(t)(t)^{-1}(t)^{-1} = 1 + t^2 - t = t^2 - t + 1.$$

If we $x = y = t$ in

$$\frac{\partial}{\partial t}[xyxy^{-1}x^{-1}y^{-1}] = x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1}$$

then we get

$$x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1}|_{x=y=t} = (t) - (t)(t)(t)(t)^{-1} - (t)(t)(t)(t)^{-1}(t)^{-1}(t)^{-1}$$

$$= t - t^2 - 1 = -t^2 + t - 1 = -(t^2 - t + 1).$$

In light of Theorem 3.5.6 (which state that Alexander polynomials for a knot computed in different ways will differ by a multiple of $\pm t^k$ for some $k \in \mathbb{Z}$), we see that these two versions of the Alexander polynomial fot the trefoil knot are consistent.. Livingston describes these observations as "a hint of things to come" in this section (see page 125).

Definition. For $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ a vector valued function with differentiable components $\vec{f} = [f_1, f_2, \ldots, f_m]$, the *Jacobian matrix* of $\vec{f}$ is the $m \times n$ matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

Note. The Jacobian matrix is used in multi-variable calculus when making a change of variables in a multiple integral; it plays the role of "du" in the change of variables. See my online notes for Calculus 3 (MATH 2110) on 15.8. Substitutions in Multiple Integrals. We use it now to give an algorithm for computing the Alexander polynomial, but with "regular" differentiation replaced with the Fox derivative.

Note 6.4.A. We claim that we can compute the Alexander polynomial of a knot with the following algorithm, which we call "Fox's Algorithm."

Step 1. Take any presentation of the group of the knot, as described in Section 5.4. Equations in Groups and the Group of a Knot. The presentation will have one more generator than relation.

Step 2. Form the Jacobian matrix using the Fox derivative of the equations in the presentation.

Step 3. Eliminate any one of the columns of the Jacobian matrix (resulting in a square matrix).

Step 4. Substitute t for all the variables in the Jacobian matrix.

Step 5. Take the determinant of the resulting matrix and this will give the Alexander polynomial.

Example 6.4.B. We apply Fox's Algorithm to the trefoil knot. For Step 1, the trefoil knot has knot group G with presentation $\langle x, y \mid xyxy^{-1}x^{-1}y^{-1}\rangle$, as seen in Example 5.4.B. For Step 2, we have the Jacobian matrix (using the computations of Example 6.4.A above)

$$\begin{aligned} J &= \left(\begin{array}{cc} \frac{\partial}{\partial x}[xyxy^{-1}x^{-1}y^{-1}] & \frac{\partial}{\partial y}[xyxy^{-1}x^{-1}y^{-1}] \end{array} \right) \\ &= \left(\begin{array}{cc} 1 + xy - xyxy^{-1}x^{-1} & x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1} \end{array} \right). \end{aligned}$$

For Step 3, we eliminate one of the columns of J (the second column here) to get:

$$J = \left(\frac{\partial}{\partial x}[xyxy^{-1}x^{-1}y^{-1}] \right) = \left(1 + xy - xyxy^{-1}x^{-1} \right).$$

For Step 4, we set $x = y = t$ to get the matrix $\left(\, t^2 - t + 1 \, \right)$. For Step 5, the determinant (since we have just a 1×1 matrix) is $\det\left(\, t^2 - t + 1 \, \right) = t^2 - t + 1$, in agreement with Example 3.5.1 (and Appendix 2).

Example 6.4.C. In Section 5.4. Equations in Groups and the Group of a Knot we saw that the arcs of the knot of Figure 5.8 can be labeled with group elements as given in Figure 5.8, where the following relations must hold (using the notation we introduced in that section):

$$yx^{-1}zxy^{-1}xyx^{-1}z^{-1}xy^{-1}xy^{-1}x^{-1} = 1, \tag{R1}$$

$$yx^{-1}y^{-1}zyxy^{-1}z^{-1}yzyx^{-1}y^{-1}z^{-1}yxy^{-1}xy^{-1}x^{-1}yx^{-1}z^{-1}xy^{-1}xyx^{-1} = 1, \tag{R2}$$

$$(z^{-1}y^{-1}zyxy^{-1}z^{-1}yz)yx^{-1}y^{-1}z^{-1}yxy^{-1} = 1 \tag{R3$'$}$$

However, by Note 5.4.A we see that we only need two of the relations in order to determine the group of the knot. So we use relations (R1) and (R3′) (since (R2) would yield a lengthier computation for the Fox derivative). If we calculate the 2×3 Jacobian matrix and then delete on column (say the column corresponding to partials with respect to y, then set $x = z = t$ we find that we get the 2×2 matrix

$$A(t) = \begin{pmatrix} -t^4 + t - 2 & -t + 1 \\ -t + 2 & 1 - 3t^{-1} + t^{-2} \end{pmatrix}$$

as is to be verified in Exercise 6.4.3. So by Fox's Algorithm, the Alexander polynomial is $A_K(t) = \det(A(t)) = -2t^2 + 10t - 15 + 10t^{-1} - 2t^{-2}$.

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