Chapter 7. Numerical Invariants

Section 7.1. Summary of Numerical Invariants

Note. In this section, we give a brief recap of the numerical knot invariants introduced so far. So there are no proofs or new definitions in this section.

Note. Every knot forms the boundary of an oriented surface by Theorem 4.3.7. The **genus** of a knot is the minimal genus that occurs among all Seifert surfaces for the knot (see Section 4.3. Seifert Surfaces and the Genus of a Knot). Only the unknot has genus 0.

Note. We explored mod p labelings of a knot diagram in Section 3.3. A Generalization of Colorability, mod p Labelings. This required the solving of a system of linear equations mod p (or equivalently, solving the system of equations in the field $\mathbb{Z}_p$). In Section 3.4. Matrices, Labelings, and Determinants we defined the determinant and the **mod p rank** of a knot (which is the dimension of the solution space of the system of linear equations mod p). In Exercise 3.4.6, it is to be shown that the number of mod p labelings of a knot of rank n is $p(p^n - 1)$.

Note. We defined the Seifert matrix of a knot in Section 6.1. The Seifert Matrix (based on the Seifert surface of a the knot). We saw in Section 6.3. Signature of a Knot, and Other S-equivalent Invariants that for V the Seifert matrix of knot, the **determinant of the knot** (defined in Section 3.4) can be computed as $|\det(V + V^t)|$ (see Note 6.3.A).

Note. In Section 6.3. Signature of a Knot, and Other S-equivalent Invariants, we defined the **signature of a knot** K, denoted $\sigma(K)$, as the signature of the symmetric matrix $V + V^t$ (where V is the Siefert matrix of the knot). It is to be shown in Exercise 6.3.6 that the signature is additive under connected sums of knots (that is, $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$.

Note. We now consider the **degree of the Alexander polynomial** of a knot. However, in light of Theorem 3.5.6 (also see Note 3.5.B), we must first "normalize" an Alexander polynomial to make this idea well-define. We normalize an Alexander polynomial by multiplying by $\pm t^k$ for some $k \in \mathbb{A}$ such that the result is a polynomial (so there are no negative powers of t) and the constant term is nonzero (so it is of minimal degree among all Alexander "polynomials" for the knot with no negative powers of t). In Exercise 6.2.5 it is to be shown that the Alexander polynomial of the connected sum of two knots is the product of their individual Alexander polynomials. So the degree of the Alexander polynomial is additive under connected sums of knots.

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