Section 7.2. New Invariants

Note. In this section, we introduce two new numerical invariants: the crossing index and the unknotting number.

Note. Recall from Section 2.4. Diagrams and Projections that a regular projection of a knot is (informally) a knot diagram where the information of over-crossings and under-crossings has been lost. Reidemeister moves 1 and 2 change the number of double points (i.e., crossings), so different projections for a given knot can have different numbers of double points. This leads to the following definition.

Definition. The least possible number of double points in a projection of knot K is the *crossing index* of the knot, denoted $C(K)$.

Note. The unknot has crossing index 0. There are no knots of crossing index 1 or 2, which is easily seen. The trefoil has a crossing index of 3. Livingston comments that computation of the crossing index is "especially difficult," that "little is known in detail" about the number of knots of a given crossing index, and that it is conjectured but unproven that the crossing index adds under the process of taking a connected sum (see page 132).

Note. Given an oriented knot diagram, it is possible to find a set of crossings such that if each is switched from right-handed to left-handed, or vice versa (or in an unoriented diagram, changing over-crossings to under-crossing, or vice versa), the knot becomes unknotted.

Definition. The switches of the crossings described above are called *crossing changes*. The minimum number of crossing changes that is required to convert a knot to the unknot, taken over all possible diagrams of the knot, is the *unknotting number* of the knot.

Note. An algorithm to convert a knot K into the unknot using crossing changes is given as follows. Consider a knot projection for K and a point p on the projection, where p is not a double point. Trace along the knot projection starting at point p. Each crossing point is reached twice. The second time it is met, make that strand go under the first strand reached passing through that crossing point. Stop when you have returned to point p. In Exercise 7.2.3 it is to be proved that this algorithm results in the unknot. Notice that this puts an upper bound on $U(K)$ of the number of crossings in a knot diagram of K. To illustrate the algorithm, consider the knot diagram given in Figure 7.1 (left) and the knot diagram that results from applying the algorithm (right). It is fairly easy to see that the diagram on the right represents the unknot. Notice that only 4 crossings are changed so that an upper bound on the unknotting number of the given knot is 4.

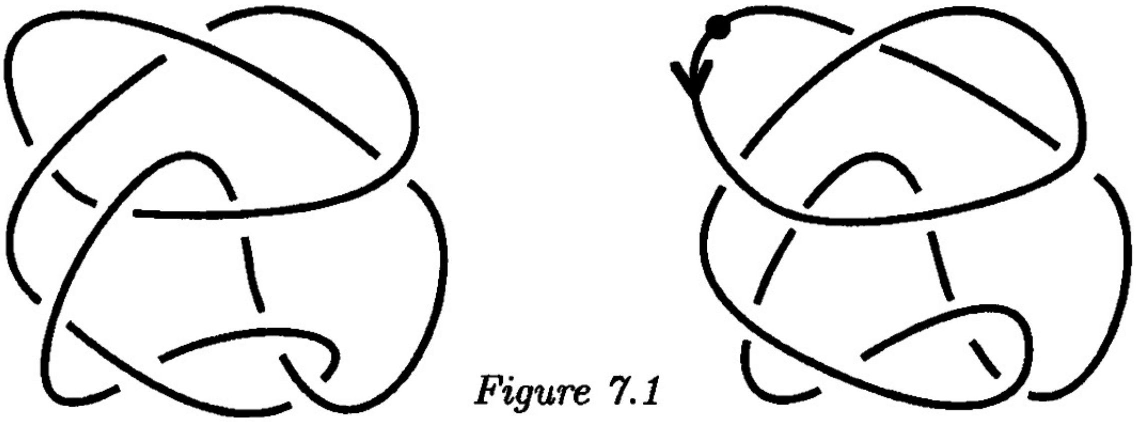

Figure 7.1

Note. Only the unknot has unknotting number 0. The n-twisted doubled knots, an example of which is given in Figure 3.6(b), demonstrates an infinite family of (nonequivalent) knots with unknotting number 1; Livingston claims that these knots have different Alexander polynomials for different values of n (see page 134).

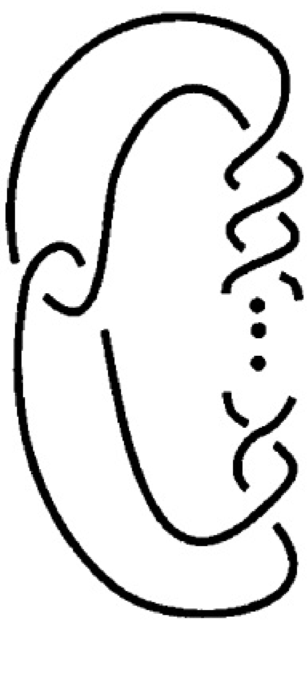

Figure 3.6 (b)

Note. It is unknown how the unknotting number is affected by taking connected sums. It is conjectured that the unknotting number is additive in this setting. One special case that has been proved is that the connected sum of two knots, each with unknotting number one, has unknotting number two (see Martin Scharlemann, Unknotting Number One Knots are Prime, *Inventiones Mathematicae* **82**, 37–56 (1985); an online copy is available through the European Digital Mathematics Library, accessed 3/9/2021)

Note. Steven Bleiler presented the two knot diagrams of the same knot given in Figure 7.2 (below). This "fascinating example" has the property that the knot diagram on the right has more crossings than the knot diagram on the left, but the two marked crossings in the right diagram can be changed and the result is the unknot. However, on the left there are no two crossing that can be changed to result in the unknot. One conclusion from this example is that the crossing number of a knot may not be based on a knot diagram with the minimum number of crossings. See Steven A. Bleiler, A Note on Unknotting Number, *Mathematical Proceedings Cambridge Philosophical Society*, **96**(3), 469–471 (1984).

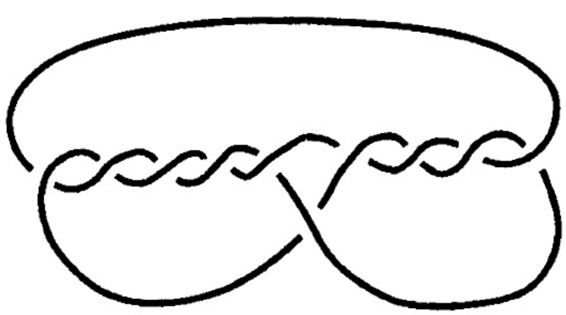

Figure 7.2