Section 7.3. Braids and Bridges

Note. In this section, we introduce braids and closed braids, and discuss the relationship between closed braids and links. We define the braid index, introduce the braid group, and define the bridge index of a knot (which is based on knot diagrams).

"Definition." An *n-stranded braid* consists of n disjoint arcs running vertically in 3-space, the set of starting points for the arcs lying directly above the set of ending points. By convention, we take the starting points lying along a horizontal line and evenly spaced. Overcrossings and undercrossings are represented diagrammatically as with knot diagrams. See Figure 7.3.

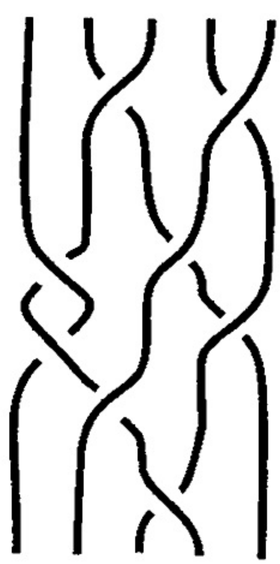


Figure 7.3

Note/Definition. We can convert a braid into a link by attaching the starting points to the ending points. The result is called a *closed braid*:

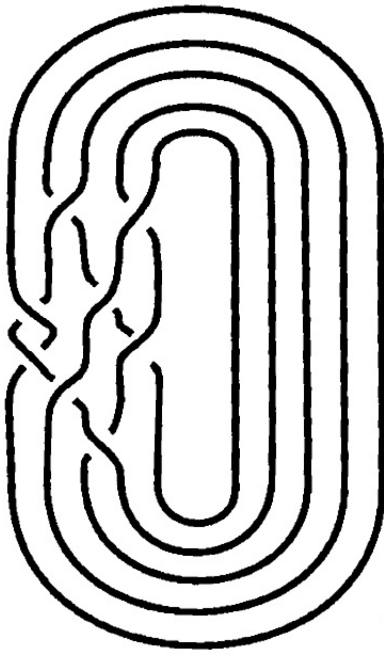


Figure 7.4

Note. Every knot and link arises from a braid by attaching starting and ending points of the braid. This result is originally due to Alexander (see James Alexander, "A Lemma on a System of Knotted Curves," *Proceedings of the National Academy of Sciences of the United States of America*, **9**(3), 93-95 (1923); a copy is available on the PNAS website [accessed 3/10/2021]). The steps by which this is done are as follows.

Step 1. Start with a polygonal knot diagram and put an orientation on it.

Step 2. Pick a point (called the *braid axis*) in the plane of the diagram which does not lie on the knot.

Step 3. We next subdivide the polygonal segments, as needed, so that we can "arrange" every segment of the polygon to run clockwise around the braid axis. If some segment runs counter clockwise, we subdivide it and "pull" it across the braid axis (using Reidemeister moves as needed for the subdivided segments interacting with the other segments).

Figure 7.5 illustrates how Step 3 is performed (see the blue segments).

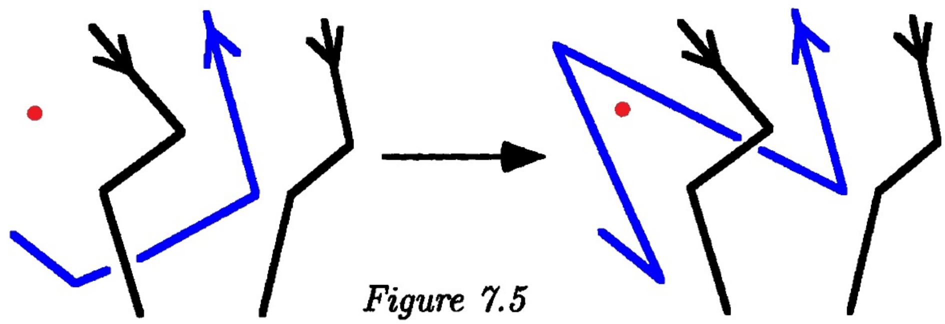
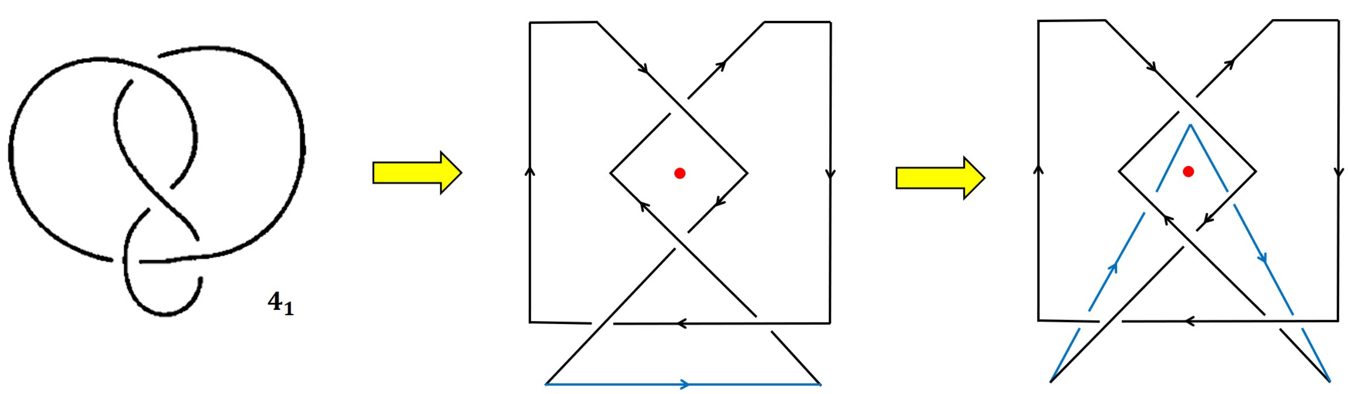
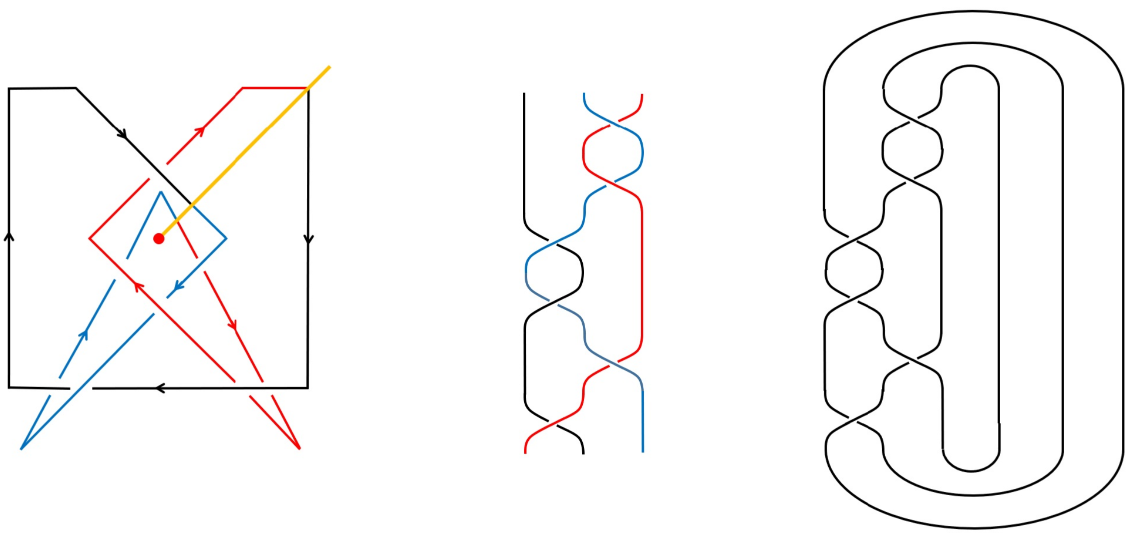


Figure 7.5

Exercise 7.3.2(a). We illustrate this process by drawing the knot 4_1 as a closed braid. For Step 1 we take the knot diagram and convert it to a polygonal not diagram with an orientation. For Step 2, we pick the braid axis (the red point) given in the figure.



For Step 3, we see that the lower (blue) arc runs counter clockwise around the braid axis, so we subdivide it and pull it across the braid axis (using undercrossings with respect to the other arcs, as illustrated). To finish, notice that if we draw a line (in orange) from the braid axis to the exterior of the knot diagram, we cross three arcs and we can rearrange the diagram to produce a 3-stranded braid.



Notice that we can use Reidemeister move 2 and ignore the two lower right crossings of the black arc over the red arc. To construct the 3-stranded braid given here, consider starting with the blue arc where it intersects the orange line. Notice that the blue arc ends where the red arc begins (when the blue arc returns to the orange line), the red arc ends where the black arc begins, and the black arc ends where the blue arc begins. Notice that we can use Reidemeister move 2 and ignore the two lower right crossings of the black arc over the red arc. So we can close the braid , as given on the right.

Note/Definition. Different braids can determine the same knot. The *braid index* of a knot K, denoted $\text{brd}(K)$, is the minimum number of strands that are required in a braid determining knot K. We can easily see that for a connected sum of knots K and J, $\text{brd}(K\#J) \leq \text{brd}(K) + \text{brd}(J)$ (that is, the braid index is "subadditive" under the connected sum). Figure 7.6 (modified some from the version in the book) illustrates this behavior. The two braids on the left form a connected sum by inserting the blue parts of the right braid in between the two parts of the right braid and leaving the red parts of the right braid connected to themselves (in the corresponding closed braid). In the resulting braid, the new connections in the closed braid are correspondingly color coded. This elementary idea shows that a connected sum of an n_1-strand braid and an n_2 stranded braid *can* be expressed as an $(n_1 + n_2)$-stranded braid (through technically, it is the closed braids for which we take the connected sum).

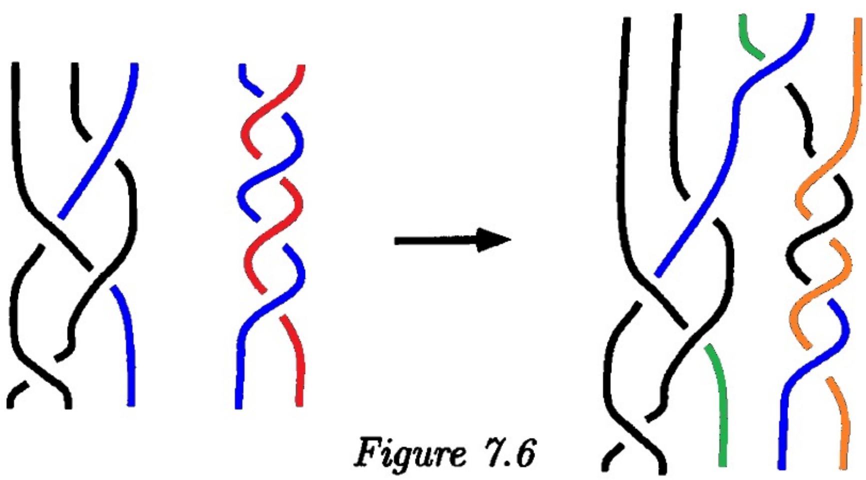


Figure 7.6

Note. According to Braid Group page, "Braid groups were introduced explicitly by Emil Artin in 1925... they were already implicit in Adolf Hurwitz's work on monodromy from 1891. Braid groups may be described by explicit presentations, as was shown by Emil Artin in 1947." Artin's 1947 work is "Theory of Braids," *Annals of Mathematics*, **48**(1), 101-126 (1947). Given two n-stranded braids, we can connect the "bottom" of one to the "top" of the other create another n-stranded braid. This is the binary operation in the *braid group* of n-stranded braids. A nice description of braid groups is given in Kunio Murasugi's *Knot Theory and Its Applications* (translated by Bohdan Kurpita), Boston: Birkhauser (1996, originally published in Japanese in 1993). See Section 10.2, "The Braid Group."

Note. Figure 7.7 illustrates a modification of an n-stranded braid to give a $(n+1)$-stranded braid which simply result from adding a new strand that has no real effect on the braid (in terms of crossings, say). Livingston call this "stabilization," but Murasugi refers to it as the "Markov move M_2." Notice that the left side of Figure 7.7 represents the original connections to be made (in blue) when the braid is closed, and the new strand is introduced on the right with the new connections (so the new strands is simply added in a way that it extends the blue strands without introducing new crossings).

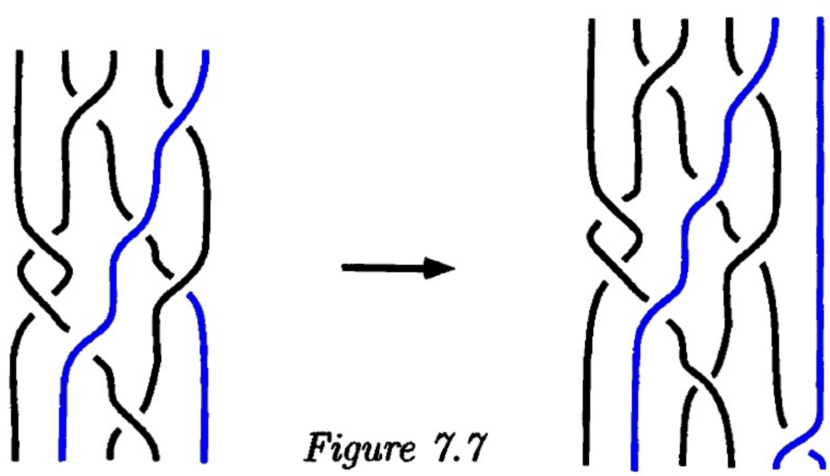


Figure 7.7

Definition. If a given n-stranded braid is multiplied by an element of the braid group "above" and the inverse of the element "below," then the result is called the *conjugate* of the original braid (in the braid group). Livingston calls the *conjugation* and Murasugi refers to it as the "Markov move M_1." If one braid can be converted into another braid by a sequence of Markov moves, M_1 and M_2, and their inverses, M_1^{-1} and M_2^{-1}, then the two braids are *Markov equivalent.*

Note. Markov's Theorem states that two braids form the same link if and only if the two braids are Markov equivalent (see Theorem 10.3.2 in Marasugi's book).

Note. Figure 7.8 gives a knot diagram where the relative maxima and relative minima are marked with large dots (where we interpret the knot diagram as a projection into the Cartesian plane and we have the relative extrema as a curve in the Cartesian plane).

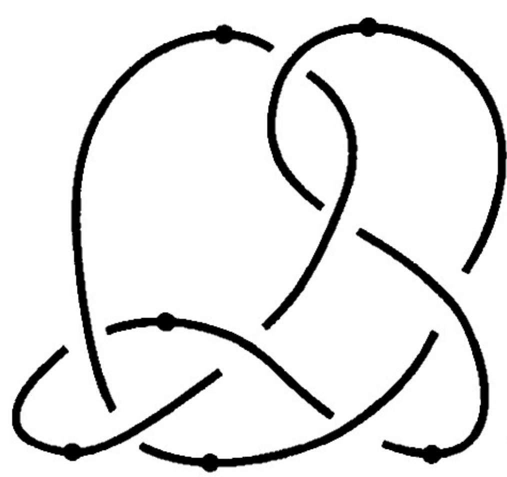


Figure 7.8

Different diagrams of a knot can have a different number of maxima.

Note. The minimum number of relative maxima taken over all possible projections of a knot K is the *bridge index* of K, denoted brg(K).

Note. For a somewhat different approach to the bridge index, see Murasugi's *Knot Theory and Its Applications* Section 4.4, "The Bridge Number." The next result (due to Horst Schubert in "Über eine numerische Knoteninvariante [About a Numerical Knot Invariant]," *Mathematische Zeitschrift*, **61**(1), 245-288 (1954)) appears as Theorem 4.3.2 in Murasugi.

Theorem 7.3.1. For knots K and J, we have

$$\mathrm{brg}(K\#J) = \mathrm{brg}(K) + \mathrm{brg}(J) - 1.$$

Note. Livingston declares the proof of Theorem 7.3.1 "quite difficult" and Murasugi omits a proof. Figure 7.9 gives the idea behind the proof. The upper knot has bridge index 2 and the lower graph has bridge index 3. In the connected sum of the two knots a relative maximum is lost from the lower knot, accounting for the -1 term in Theorem 7.3.1.

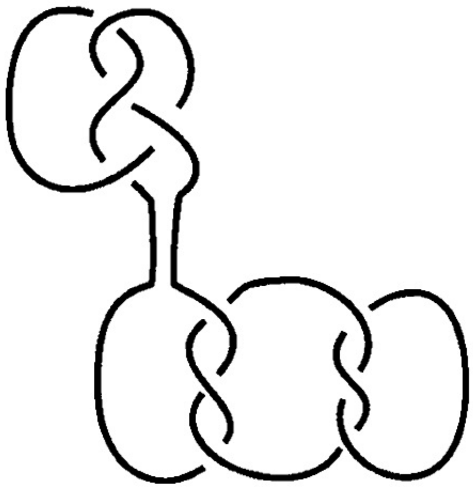


Figure 7.9

In Exercise 7.3.3 it is to be shown that every knot with bridge index 2 is prime.

Revised: 3/11/2021