Section 7.4. Relations between Numerical Invariants

Note. In this section, we relate several of the knot parameters we have introduced, including the crossing number, the genus, the bridge index, the mod p rank, the signature, and the unknotting number.

Note. In Section 4.3. Seifert Surfaces and the Genus of a Knot we gave an algorithm by which a Seifert surface can be created for any given knot (see the proof of Theorem 4.3.7). We defined the genus of a knot and introduced the idea of a "Seifert circle" (also in the proof of Theorem 4.3.7). In Exercise 4.3.4 it is to be shown that the number of Seifert circles s, **the genus g, and the crossing number** cr are related by the equation $2g = cr - s + 1$.

Note. In Section 3.3. A Generalization of Colorability, mod p Labelings we described labelings of the arcs of a knot with the elements of $\mathbb{Z}_p$. Livingston claims that any mod p labeling of a knot with bridge index n is determined by the labels on the n "top" arcs and that there can be at most an n-dimensional space of labelings. One labeling is the trivial one where all edges are labeled with 0, so **the mod p rank of the knot** (defined in Section 3.4. Matrices, Labelings, and Determinants) is at most **the bridge index** minus 1, $\text{brg}(K) - 1$.

Note. In Section 6.3. Signature of a Knot, and Other S-equivalent Invariants we defined the signature of a knot K, $\sigma(K)$, as the signature of matrix $V + V^T$ (where V is the Seifert matrix of K). We just defined the unknotting number, $U(K)$, in Section 7.2. New Invariants. Livingston claims that **the unknotting number and the signature** are related as $2U(K) \geq |\sigma(K)|$. He gives an argument for this on pages 142 and 143, but we take this as given.

Note. Livingston claims that the **unknotting number of a knot is greater than or equal to the mod p rank** of the knot, for all p, but does not give an argument for this (see page 143). He does give an argument for the following results, but take this as given, also.

Corollary 7.4.A. If K is a knot with unknotting number 1, then the mod p rank of K is at most 1.

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