Section 7.5. Independence of Numerical Invariants

Note. In this section, we give several pairs of numerical invariants which are not related to each other. We support our claims by examples. We take Livingston's quantitative claims as true without computational justification.

Note. The **bridge index** of Section 7.3 is unrelated to the **degree of the Alexander polynomial**. The $(2, n)$-torus knots (see Chapter 1. A Century of Knot Theory) each have bridge index 2 but can have arbitrarily high degree Alexander polynomials.

Note. The **mod p rank can vary for different values of p** (see Section 3.4. Matrices, Labelings, and Determinants). For example, the connected sum of k trefoil knots and j 5-twist knots has has mod 3 rank k and mod 5 rank j. Given any finite set of primes, similar examples can be constructed showing the independence of mod p ranks.

Note. The $(2, n)$-torus knot has signature $n-1$ (as to be shown in Exercise 6.3.9), and it has a bridge index of 2. So **no bound on the signature can be based on the bridge index**.

Note. The $(2, n)$-torus knot has **bridge index** 2 and arbitrarily high **unknotting number**. A knot can be "doubled" as shown in Figure 7.12 for the trefoil knot, gives knots with unknotting number 1 knots of large bridge index.

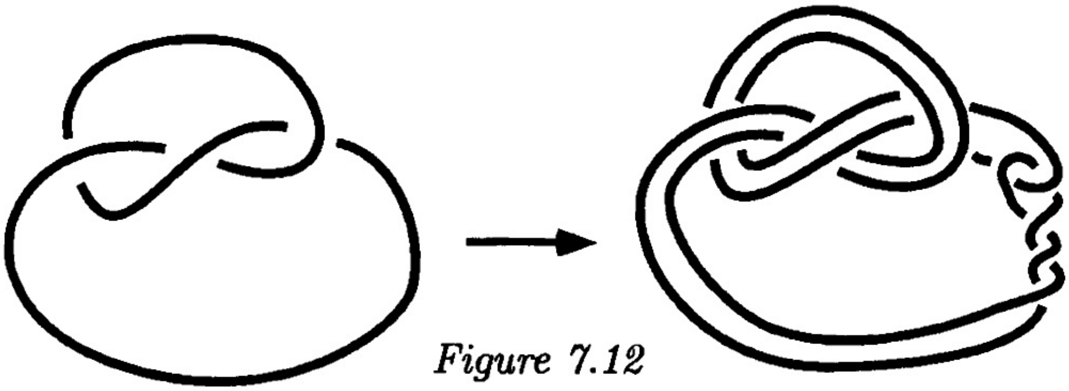

Figure 7.12

Horst Schubert proved that doubling a knot double the bridge index, except in one special case (which is to be explored in Exercise 7.5.3).

Note. The next theorem relates the bridge index to a labeling with elements of the symmetry group S_n.

Theorem 7.2. If a knot K can be labeled with transpositions which generate S_n, then $\text{brg}(K) \geq n$.

Note 7.5.A. We now describe an application of Theorem 7.2. Suppose that a knot diagram has been consistently labeled with 3-cycles from S_n. Notice that the set of all 3-cycles does not generate S_n, but only the alternating group A_n (See Exercise 15.39(b) of John Fraleigh's *A First Course in Abstract Algebra*, 7th edition [Addison Wesley, 2003]; see also my online notes for Introduction to Modern Algebra [MATH

4127/5127] on Supplement. The Alternating Groups A_n are Simple for $n \geq 5$, or Lemma I.6.11 in my notes for Modern Algebra 1 [MATH 5410] on Section I.6. Symmetric, Alternating, and Dihedral Groups). We claim that this labeling leads to a consistent labeling of some double knot using transpositions. The procedure involves taking the 3-cycle label (a, b, c) on a bridge (i.e., an arc containing a "local maximum") of the original knot and labeling the two corresponding arcs in the doubled knot with (a, b) and (a, c) (notice that $(a, b, c) = (a, b)(a, c)$; we multiply transpositions from left to right). The consistency condition of Section 5.2. Knots and Groups then determines the rest of the labelings of the two knots which result from the doubling. Livingston claims (page 147) that any problem with consistency at the bottom can be corrected by adding twists and leaves confirmation of this as Exercise 7.5.4

Note 7.5.B. A set of permutations P in S_n is *transitive* if for any $i, j \in \{1, 2, \ldots, n\}$ there is $\pi \in P$ such that $\pi(i) = j$. We claim that if the set of 3-cycle labels of Note 7.5.A form a transitive set of permutations, then the transpositions that are used in labeling the doubled knot will generate all of S_n (the proof of which is "left to the reader"). In Exercise 7.5.5 it is to be shown that the connected sum of k $(2, 5)$-torus knots can be consistently labeled with a transitive set of 3-cycles from S_{3+2k}. Therefore, there exists knots which can be labeled with transpositions which generate S_n (namely, the knots that result from the knot doubling). So by Theorem 7.5.2, there exist doubled knots of large bridge index (namely, at least $2k + 3$ for any given $k \in \mathbb{N}$).

Note. The $(2, n)$-torus knots have bridge index 2 and arbitrarily high genus. So **no bound on the genus can be based on the bridge index**.

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