Section 8.1. Amphicheiral and Reversible Knots

Note. In section Section 2.5. Orientations we defined the reverse K^r of an oriented knot K. Changing all of the crossings of knot K (that is, interchanging over crossing and under crossings) yields knot K^m. This idea was introduced in Exercise 2.5.6, in which it was to be shown that K^m is equivalent to the mirror image of K corresponding to the reflection of its diagram through the y-axis (treated as the vertical axis) of the knot diagram. In this section these knots are used to define certain knot symmetries.

Definition. An oriented knot K is *reversible* if K is oriented equivalent to K^r. It is *positive amphicheiral* if it is oriented equivalent to K^m, and *negative amphicheiral* if it is oriented equivalent to K^{rm}.

Note 8.1.A. In Figure 8.2, a diagram for the oriented knot 4_1 (the "figure-8 knot") is given on the left, and a diagram that results from rotating the knot about the y-axis (treated as the vertical axis) through 180° is given on the right (notice that it is not the 2-dimensional *diagram* that is rotated about the y-axis, but instead the 3-dimensional *knot* itself that is rotated). Ignoring the orientations, the diagrams are the same, but when we consider the orientations we see that one is the reverse of the other. So this knot is reversible. We can change all the crossings and easily confirm that the resulting knot is the equivalent to 4_1, so that this knot is positive amphicheiral. In Exercise 8.1.1 it is to be shown that for reversible knots, being positive amphicheiral is equivalent to being negative amphicheiral. So the knot 4_1, being reversible and positive amphicheiral is also negative amphicheiral.

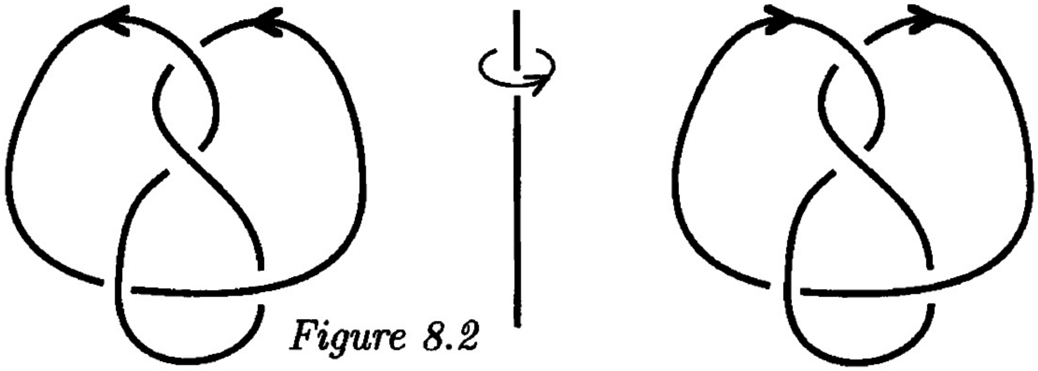


Figure 8.2

Note. Figure 8.3 (modified as given below) gives an oriented diagram of the $(3,5,3)$-pretzel knot on the left, and this same diagram rotated 180° about an axis perpendicular to the plane of the diagram (i.e., the page) on the right. Notice that this results in the same diagram, but with the orientation reversed. So this knot is reversible.

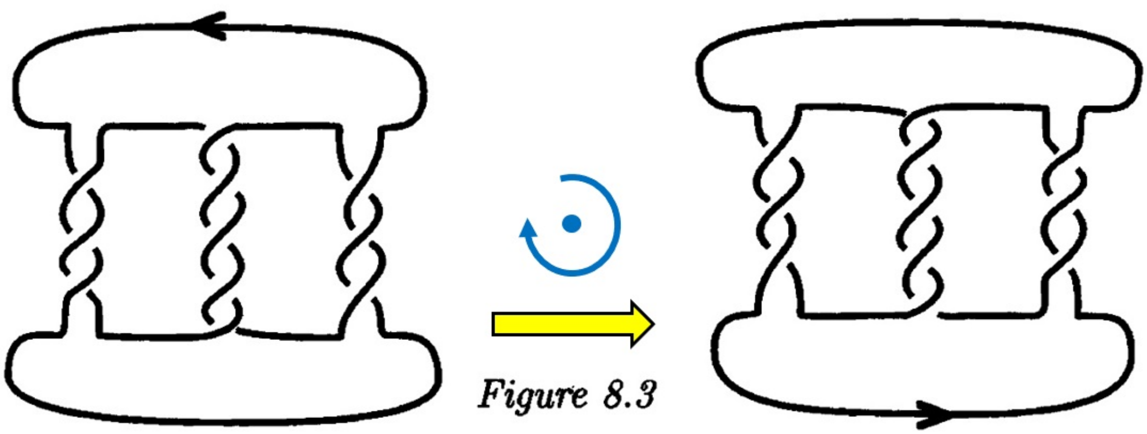


Figure 8.3

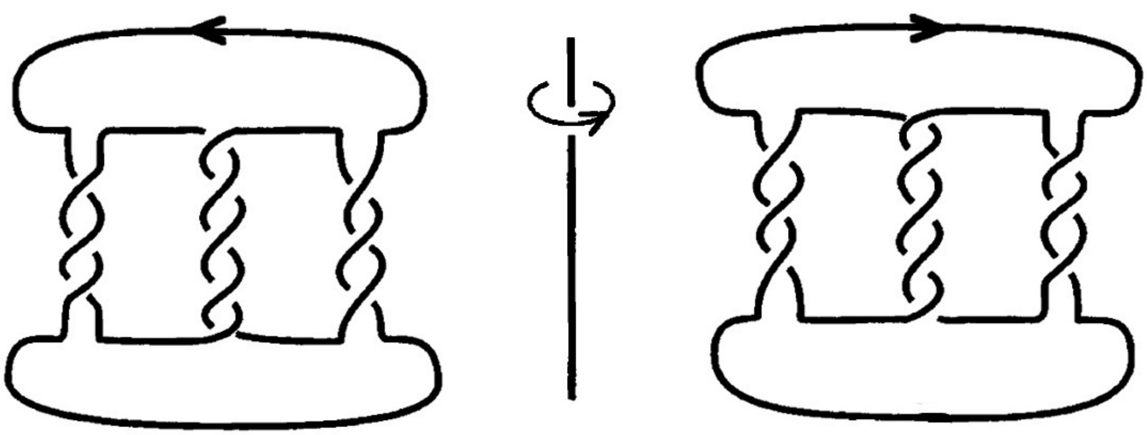
Livingston claims that the only reversible pretzel knots are those with two of the bands having an equal number of twists and that a signature calculation shows that this pretzel knot is neither positive nor negative amphicheiral (see page 153).

Note. The two knots given above (in Figures 8.2 and 8.3) are reversible, as is demonstrated by either a rotation about the y-axis or the z-axis. These are examples of "strongly reversible" knots, as defined next. Not all reversible knots have a symmetry that is so easily recognized.

Definition. A knot is *strongly reversible* if it is equivalent to a knot that is carried to its reverse by either a $180°$ rotation about the y-axis, or reflection through the (y, z)-plane.

Note. Notice that a rotation about the y-axis through $180°$ changes all the crossings in a diagram.

Note. We saw in Figure 8.2 that the 4_1 knot is strongly reversible. We can see that the $(3, 5, 3)$-pretzel knot is reversible by rotating the knot $180°$ about the y-axis, as follows:



So the $(3, 5, 3)$-pretzel knot is strongly reversible as well.

Note. The connected sum of the left-handed and right-handed trefoil knots of Figure 8.4 is not invariant under a 180° rotation about the y-axis, as illustrated in the following modified version of Figure 8.4:

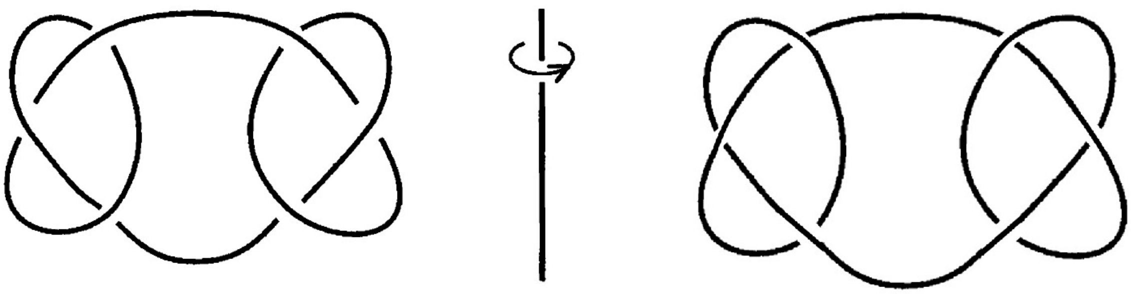


Figure 8.4

However, when we reflect this knot through the (y, z)-plane we "clearly" get the same diagram. Think of the (y, z)-plane running through the center of the knot vertically; since the knot is the connected sum of the left-handed trefoil (on the left) and the right-handed trefoil knot (on the right) and these knots are mirror images of each other by Note 1.A, then the connected sum has this symmetry.

Note. We conclude this section with "strongly" definitions of amphicheirality.

Definition. A knot K is *strongly positive amphicheiral* if there is a self-map T of 3-space (that is, a continuous bijection with a continuous inverse that maps 3-space onto itself) with T^2 the identity mapping on 3-space, such that $T(K) = K^m$. Similarly, K is *strongly negative amphicheiral* if there is such a T with $T(K) = K^{rm}$.