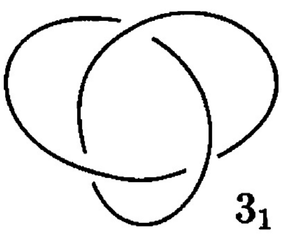
# **Section 8.2.** Periodic Knots

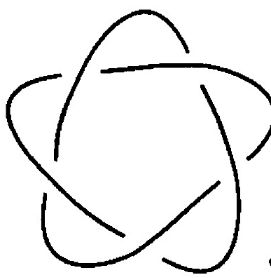
**Note.** In this section, we define a periodic knot as one for which there is a diagram which has a rotational symmetry. We describe quotient knots and covering links, and define the linking number of a knot diagram. We prove a theorem concerning the connect components of covering links.

**Definition.** A knot $K$ is *periodic* with period $q$ if $K$ has a diagram which misses the origin and which is carried to itself by a rotation of $(360/q)^\circ$ about the origin.

**Note.** The knot $3_1$ is periodic with period 3, knot $5_1$ is periodic with period 5, and knot $7_1$ is periodic with period 7 (see the figure below, which includes images from Appendix 1).



$3_1$



$5_1$

$7_1$

Figure 8.6 gives another knot diagram for knot $5_1$, from which we see that it also is periodic with period 2.

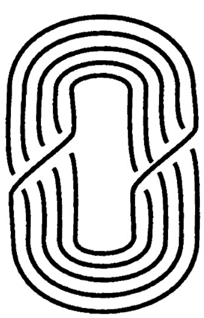


*Figure 8.6*

**Note/Definition.** For a periodic diagram of a knot, we describe the *quotient knot*. Roughly, for a period $q$ knot diagram, a "sector" of the knot of size $(360/q)^\circ$, closed off similar to the way a closed braid closes off a braid, forms a quotient knot diagram. The quotient knot then gives a fundamental piece of the periodic knot, and the periodic knot can be reproduced from the quotient knot by pasting together $q$ copies of the quotient knot. Figure 8.7 below illustrates the quotient knot in two cases; notice that on the left of Figure 8.7, the quotient knot is in fact the unknot.

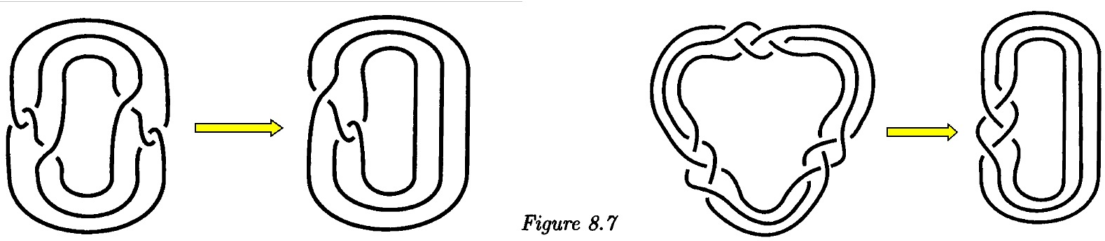


*Figure 8.7*

**Note/Definition.** We now describe a process that reverses the process of finding a knot quotient from a periodic knot diagram. Given a knot diagram that misses the origin and given an integer $q \geq 2$, we construct a knot (or link) having the original knot as a quotient. In Figure 8.8 a given knot (left) is used to produce a *covering link* (right) which is periodic with period 4. which has the knot on the left as a quotient knot.

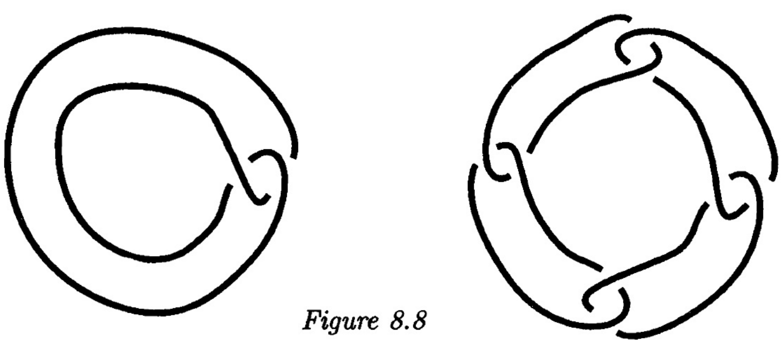
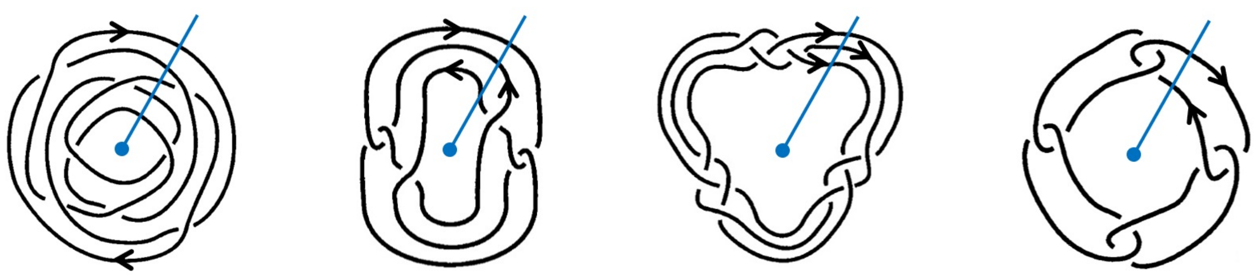


*Figure 8.8*

**Note.** For a given oriented knot diagram which does not include the origin, pick a ray from the origin such that none of the points of intersection of the ray and knot are tangential. The intersection number is the number of intersection points at which the knot crosses the ray in the left-hand sense (i.e., in the clockwise direction) minus the number of right-hand sense (i.e., in the counterclockwise intersections). The *linking number* of the diagram with the $z$-axis, denoted $\lambda$, is the absolute value of the intersection number of the knot with the ray.

**Note.** The periodic knot diagram in Figure 8.1 has linking number $\lambda = 5$. The periodic knots in Figure 8.7 have linking number $\lambda = 1$ and $\lambda = 3$. The linking number of the link in Figure 8.8 (either the period 4 link or the quotient knot) is $\lambda = 0$, as illustrated in the following modifications of the figures:



$\lambda = 5$ $\quad$ $\lambda = 1$ $\quad$ $\lambda = 3$ $\quad$ $\lambda = 0$

**Note.** By Exercise 8.2.6, if a knot diagram is periodic then the linking number of the knot with the $z$-axis is the same as the linking number of the quotient with the $z$-axis. The converse also holds; that is, if a periodic diagram for a knot arises from the covering construction, then the linking number of the periodic diagram is the same as the linking number of the original knot. The next theorem gives conditions under which the covering link is a knot (as opposed to a link; in Figure 8.7 [left] where the original quotient knot is a knot, but the covering construction with period 3 is only a link and not a knot).

**Theorem 8.2.1.** If a knot diagram for $K$ misses the origin, the corresponding $q$-fold covering link $L$ has a single component if the linking number is relatively prime to $q$. More generally, the number of complements in $L$ is the greatest common divisor of the linking number $\lambda$ and $q$.

**Note.** Different periodic diagrams for a given knot can have different linking numbers. The trefoil knot $3_1$ has a periodic knot diagram of period 3 (the "standard" diagram for $3_1$) and linking number 2. Figure 8.11 gives a periodic diagram of the trefoil knot with period 2 and linking number 3. We'll see in the next section that if two periodic diagrams of a given knot have the same period then they will have the same linking number.

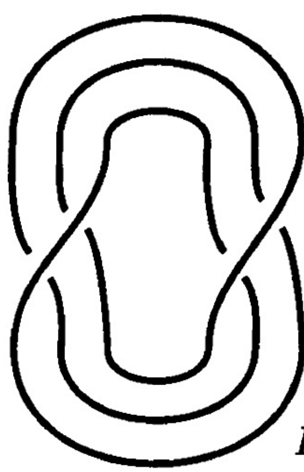


*Figure 8.11*