Section 8.3. The Murasugi Conditions

Note. Kunio Murasugi gave criteria for testing a knot for possible periods. This work was given in "On Periodic Knots," *Commentarii Mathematici Helvetici*, **46**, 162–174 (1971); this is available online through The European Digital Mathematics Library. The result is based on the Alexander polynomial of the knot and the Alexander polynomial of its quotient knot.

Theorem 8.3.2. Murasugi Conditions.

Suppose that a knot K has period $q = p^r$, with p prime. Let J denote the quotient knot of a period q diagram of K, and let λ be the linking number of J with the axis.

(1) The Alexander polynomial of J, $A_J(t)$, divides the Alexander polynomial of K, $A_K(t)$.

(2) The following mod p congruence holds for some integer i:

$$A_K(t) = \pm t^i (A_J(t))^q (1 + t + t^2 + \cdots + t^{\lambda-1})^{q-1} (\text{mod } p).$$

Note. Livingston gives the idea of the proof of Theorem 8.3.2 (see page 163). Now consider the quotient knot J (from which the periodic knot will be produced by "lifting"). Arrange knot J as given in Figure 8.13 where λ "cycles" have been introduced so that the linking number is λ (choose the origin to be in the center of the "cycles"). Then lift the diagram to produce a period q knot. What we can conclude from part (2) of the Murasugi Condition (Theorem 8.3.2) is that *any* period q knot with quotient knot J and with the same linking number λ will have the same polynomial as the example based on Figure 8.13 modulo p.

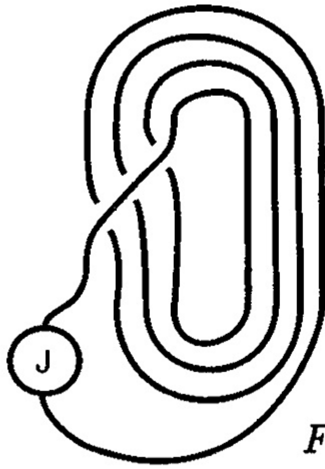
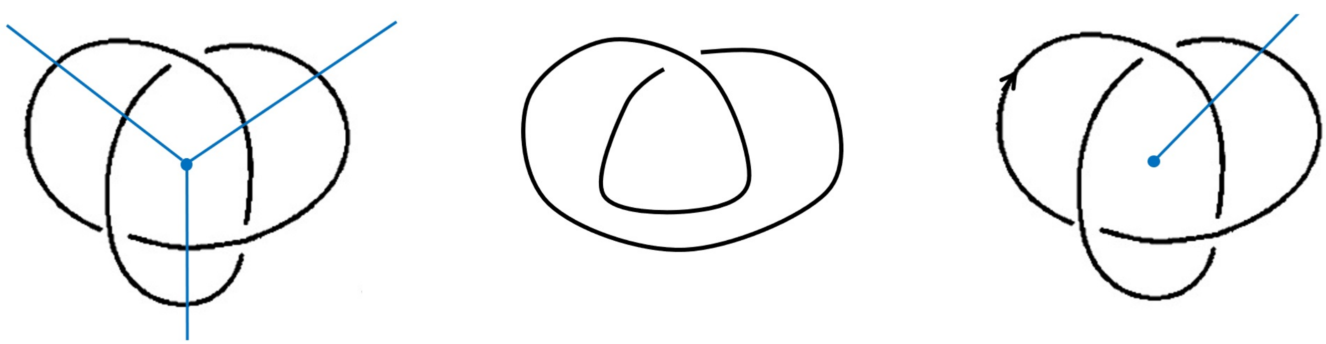


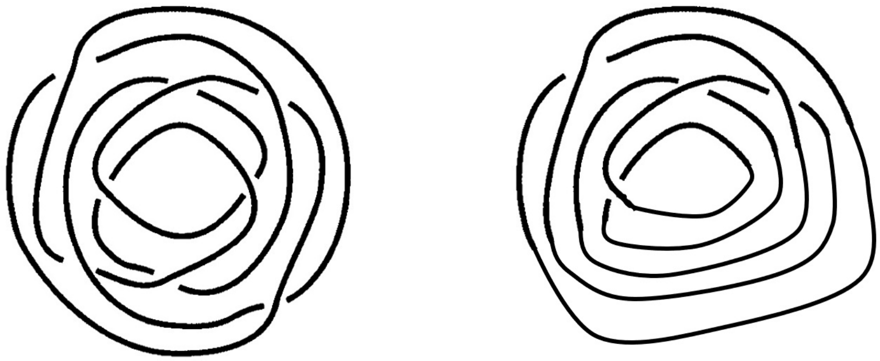
Figure 8.13

Example. The trefoil knot K has period $p = q = 3$ (left), quotient knot J of the unknot (center), and linking number $\lambda = 2$ (right):



The Alexander polynomial of the quotient knot is $A_J(t) = 1$ and the Alexander polynomial of the trefoil knot is $A_K(t) = t^2 - t + 1$, so $A_J(t)$ divides $A_K(t)$ and (1) of the Murasugi Conditions (Theorem 8.3.2) is satisfied. Also, $(A_J(t))^q(1+t+t^2+\cdots+t^{\lambda-1})^{q-1} = (1)(1+t^{(2)-1})^{(3)-1} = (1+t)^2 = t^2+2t+1$ and $t^2+2t+2 \equiv t^2-t+2$ (mod 3); that is, $A_J(t) \equiv A_K(t)$ (mod p). So (2) of the Murasugi Conditions is satisfied.

Example. Figure 8.1 gives a period $p = q = 2$ diagram of knot $K = 7_6$ (see below, left), with quotient knot J of the unknot (right), and linking number $\lambda = 5$:



The Alexander polynomial of the quotient knot is $A_J(t) = 1$ and the Alexander polynomial of the knot $K = 7_6$ is $A_K(t) = t^2 - t + 1$ (see Appendix 2), so $A_J(t)$ divides $A_K(t)$ and (1) of the Murasugi Conditions (Theorem 8.3.2) is satisfied. Also, $(A_J(t))^q(1+t+t^2+\cdots+t^{\lambda-1})^{q-1} = (1)(1+t+t^2+t^3+t^{(5)-1})^{(2)-1} = t^4+t^3+t^2+t+1$ and $t^4 + t^3 + t^2 + t + 1 \equiv t^4 - 5t^3 + 7t^2 - 5t + 1 \pmod 2$; that is, $A_J(t) \equiv A_K(t) \pmod p$. So (2) of the Murasugi Conditions is satisfied.

Note. As opposed to dealing with Alexander polynomials which are "equal modulo p," we could treat these as polynomials over the field $\mathbb{Z}_p$. That is, we consider the ring of polynomials $\mathbb{Z}_p[x]$ (see my online notes for Introduction to Modern Algebra [MATH 4127/5127] on Section IV.22. Rings of Polynomials). We now consider some definitions and results from Section IV.23. Factorizations of Polynomials over a Field. First, if f is a d_1 degree polynomial in $\mathbb{Z}_p[x]$ and g is a d_2 degree polynomial in $\mathbb{Z}_p[x]$, then the product fg is a degree $d_1 + d_2$ degree polynomial in $\mathbb{Z}_p[x]$ (this follows from the fact that we multiply polynomials in $\mathbb{Z}_p[x]$ in the usual way and there are zero divisors in $\mathbb{Z}_p$). A nonconstant polynomial $f(x) \in F[x]$ is *irreducible* if $f(x)$ cannot be expressed as a product $g(x)h(x)$ of two polynomials $g(x)$ and $h(x)$ in $F[x]$ both of lower degree then the degree of $f(x)$ (see Definition 23.7 in the Section IV.23 notes). Every nonconstant polynomial $f(x) \in F[x]$ can be factored in $F[x]$ into a product of irreducible polynomials, the irreducible polynomials being unique except for order and for unit (that is, nonzero constant) factors in F (see Theorem 23.20 in Section IV.23). We now consider applications of Theorem 8.3.2 to put constraints on the period of a knot.

Example. The trefoil knot has period 3 as illustrated above, but it also has a diagram that reveals a period 2 as well (see Figure 8.11 at the end of the last section). We now show that these are the only periods of the trefoil. First, suppose the trefoil has period $q = p^r$ with p prime. The second Murasugi condition (Theorem 8.3.2(2)), states that

$$A_K(t) = \pm t^i (A_J(t))^q (1 + t + t^2 + \cdots + t^{\lambda-1})^{q-1} \pmod{p},$$

for some integer i, where $A_K(t)$ is the the Alexander polynomial of the trefoil knot K, and $A_J(t)$ is the Alexander polynomial of the quotient knot J. With $q > 3$, if $\lambda = 1$ and $A_J(t)$ is a constant, then the degree of $A_K(t)$ is 0. With $q > 3$, if $\lambda = 1$ and $A_J(t)$ is nonconstant (and so of degree at least 1 so that $(A_J(t))^q$ is degree at least q), then the degree of $A_K(t)$ is at least $q > 3$. With $q > 3$, if $\lambda \geq 2$ then the degree of $A_K(t)$ is greater than or equal to 3 (because of the $(1+t+t^2+\cdots+t^{\lambda-1})^{q-1}$ term). But each of these cases yield a contradiction since $A_K(t) = t^2 - t + 1$ is of degree 2 (see Section 7.1 for the definition of an Alexander polynomial). If q is a different composite number (in which case Theorem 8.3.2 does not apply), notice that if a diagram for a knot is of period q, then it is also of period q' for all divisors q' of q. So the only possible composite period of the trefoil knot is 6. But a period 6 diagram is also a period $p = q = 3$ diagram. But then $q - 1 = 2$ and

$$A_K(t) = t^2 - t + 1 = \pm t^i (A_J(t))^3 (1 + t + t^2 + \cdots + t^{\lambda-1})^2 \pmod 3$$

implies that $A_J(t)$ is degree 0 and that $\lambda = 2$. A period 6 diagram is also a period $p = q = 2$ diagram. But then $q - 1 = 1$ and

$$A_K(t) = t^2 - t + 1 = \pm t^i (A_J(t))^2 (1 + t + t^2 + \cdots + t^{\lambda-1})^1 \pmod 2$$

We just saw that $A_J(t)$ is degree 0, so we need $\lambda = 3$ here. But now we need both $\lambda = 2$ and $\lambda = 3$, and since this cannot hold then we have the trefoil knot does not have period 6. That is, the only periods of the trefoil are 2 and 3.

Example. Consider the knot 9_{42} (from Appendix 1):

$\mathbf{9_{42}}$

The Alexander polynomial (from Appendix 2) is $A_K(t) = t^4 - 2t^3 + t^2 - 2t + 1$. We now use Murasugi's Conditions (Theorem 8.3.2) to show that 9_{42} is not a periodic knot. If a diagram for a knot is of period q, then it is also of period q' for all divisors q' of q, so it suffices to prove that 9_{42} has no prime periods, p. Murasugi's

Condition 2 states that

$$A_K(t) = \pm t^i (A_J(t))^q (1 + t + t^2 + \cdots + t^{\lambda-1})^{q-1} \pmod{p}.$$

Since $A_K(t)$ is degree 4, then we must have $p = q \leq 5$ (or else either $(A_J(t))q$ or $(1 + t + t^2 + \cdots + t^{\lambda-1})^{q-1}$ is of degree more than 4). For $p = q = 5$, we must have $A_J(t)$ of mod 5 degree 0 and $\lambda = 2$ (so that $(1 + t + t^2 + \cdots + t^{\lambda-1})^{q-1} = (1 + t)^4$ is of degree 4). But $(t + 1)^4 = t^4 + 4t^3 + 6t^2 + 4t + 1 = t^4 + 4t^3 + t^2 + 4t + 1 \pmod{5} \neq A_K(t)$, so this cannot be the case and hence $p \leq 3$.

For $p = q = 3$, $A_K(t) = t^4 - 2t^3 + t^2 - 2t + 1 = t^4 + t^3 + t^2 + t + 1 \pmod{3}$. Notice that none of 0, 1, 2 is a mod 3 root of $A_K(t)$ so (by the Factor Theorem in field $\mathbb{Z}_3$; see Corollary 23.3 in my online notes for Intro to Modern Algebra [MATH 4127/5127] on Section IV.23. Factorizations of Polynomials over a Field) $A_K(t)$ has no linear factors. As Livingston states, “a more careful check shows that it has no quadratic factors in a mod 3 factorization” (see page 165; notice that there are a limited number of mod 3 quadratics which are potential factors). With $q = 3$ we have $q - 1 = 2$ and so it is impossible to write $A_K(t)$ as a product of the form $\pm t^i (A_J(t))^q (1 + t + t^2 + \cdots + t^{\lambda-1})^{q-1} \pmod{3}$ (regardless of what $A_J(t)$ is). So Murasugi's Condition 2 is not satisfied and so knot 9_{42} cannot have period $p = q = 3$.

For $p = q = 2$, $A_K(t) = t^4 - 2t^3 + t^2 - 2t + 1 = t^4 + 2t^3 + 3t^2 + 2t + 1 \pmod{2} = (t^2 + t + 1)^2 \pmod{2}$. Notice that $t^2 + t + 1$ is irreducible modulo 2 (since neither 0 nor 1 is a mod 2 root). With $q = 2$ we have $q - 1 = 1$ and so Murasugi's Condition 2 implies $A_K(t) = \pm t^i (A_J(t))^2 (1 + t + t^2 + \cdots + t^{\lambda-1}) \pmod{2}$. Since $A_K(t) = t^4 - 2t^3 + t^2 - 2t + 1 \neq t^4 + t^3 + t^2 + t + 1 \pmod{2}$, then we must have $(A_J(t))^2 = (t^2 + t + 1)^2$, $\lambda = 1$, and $A_J(t) = t^2 + t + 1 \pmod{2}$. However, $A_K(t)$ has roots $(1+\sqrt{2}-\sqrt{2\sqrt{2}-1})/2$, $(1+\sqrt{2}+\sqrt{2\sqrt{2}-1})/2$, $(1-\sqrt{2}-i\sqrt{2\sqrt{2}+1})/2$, and $(1-\sqrt{2}+i\sqrt{2\sqrt{2}+1})/2$ as we can verify by factoring or from a computer algebra system (notice that this implies that it is irreducible over $\mathbb{Q}$). But then we see that $A_J(t)$ does not divide $A_K(t)$ in violation of Murasugi's Condition 1. Therefore knot 9_{42} has no primer period and hence is not a periodic knot.

Note. A definition which simplifies our application of the Murasugi conditions is given in the Exercises of this section, as follows.

Definition. The *total degree* of a polynomial is the difference between the degrees of the highest and lowest degree nontrivial terms. The *mod p total degree* is the difference between the degrees of the highest and lowest terms having coefficients not divisible by p.

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