Section 8.4. Periodic Seifert Surfaces and Edmond's Theorem

Note. In this section we consider how Seifert surfaces of periodic knots are themselves periodic (the surface itself is called "equivariant"). In the Riemann-Hurwitz Formula (see Theorem 8.4.4) we relate the genus of a equivariant Seifert surface to the genus of its associated quotient knot. In Edmond's Conditions (see Corollary 8.4.6) we relate the genus of a periodic knot to its period.

Note. We saw Seifert's algorithm for constructing a Seifert surface for a given knot in Section 4.3. Seifert Surfaces and the Genus of a Knot; see Theorem 4.3.7. When applied to a periodic knot, this algorithm produces a surface that has the same periodic symmetry as the knot. As opposed to calling the surface "periodic," we have the following definition.

Definition. A Seifert surface of a periodic knot is *equivariant.* A Seifert surface F is *equivariant of period* q if it can be deformed so that so that $R_q(F) = F$ for a rotation R_q is a rotation about the z-axis of $(360/q)^\circ$.

Note. The next result is due to Allan Edmonds. It appears in his "Least Area Seifert Surfaces and Periodic Knots," *Topology and its Applications* **18**, 109–113 (1984). This is available online through ScienceDirect; see Theorem 4 in the paper.

Theorem 8.4.3. If a knot K is of period q, then there exists a period q equivariant Seifert surface, F, for K with genus$(F) = g(K)$.

Note. Recall that the genus of a knot is the minimum possible genus of a Seifert surface for the knot. Livingston states that "Deep analytic results along with topological arguments imply that among all least genus Seifert surfaces for the knot there is one of least area. Edmonds proved that such an area minimizing surface is equivariant." See page 168. The existence of a minimum area Seifert surface is given as Proposition 1 in his 1984 paper referenced above.

Theorem 8.4.4. Riemann-Hurwitz Formula.
Let F be a genus g oriented surface which is equivariant with respect to a rotation about the z-axis of angle $(360/q)^\circ$, and let G be the quotient of F. If both F and G have one boundary component, then

$$\text{genus}(F) = q(\text{genus}(G)) + (q-1)(\Lambda - 1)/2,$$

where Λ is the number of points of intersection of F (or G) with the z-axis.

Note. We give another result on linking numbers before we consider an example.

Theorem 8.4.5. If the linking number of a periodic diagram for knot K is λ then an equivariant Seifert surface for K intersects the z-axis in Λ points, where $\Lambda \geq \lambda$ and $\Lambda = \lambda \pmod 2$.

Note. Livingston gives an argument for the proof of Theorem 8.4.5 (see pages 170 and 171). We take the result as true.

Example. Consider surface F with genus$(F) = 3$, with one boundary component, and suppose that F is periodic of period q. Suppose the quotient knot is G and genus$(G) = g_G$. By the Riemann-Hurwitz Formula (Theorem 8.4.4), we have $3 = q\, g_G + (q-1)(\Lambda - 1)/2$. We can put certain restrictions on the possible periods of the surface (that is, on parameter q). If $q > 3$ then we must have $g_G = 0$. Then we have $3 = (q-1)(\Lambda-1)/2$, and we must have $(q-1)(\Lambda-1) = 6$ (in which case q and Λ are each at most 7). We then have $q \in \{4,5,6,7\}$ and $\Lambda \in \{2,3,4,5,6,7\}$. The only values of a and Λ that work are (1) $q = 4$, $\Lambda = 3$, and (2) $q = 7$, $\lambda = 2$. If $q = 3$ then the Riemann-Hurwitz Formula implies that $3 = 3g_G + (\Lambda - 1)$ and so we must have (3) $g_G = 1$, $\Lambda = 1$, or (4) $g_G = 0$, $\Lambda = 4$. If $q = 2$ then the Riemann-Hurwitz Formula implies that $3 = 2g_G + (\Lambda-1)/2$ and so we must have (5) $g_G = 1$, $\Lambda = 3$, or (6) $g_G = 0$, $\Lambda 7$. So the only possible values of the period of surface F are $q \in \{2,3,4,7\}$. So by Theorem 8.4.3, the only possible periods of a genus 3 knot are 2, 3, 4, and 7.

Note. A converse of Theorems 8.4.4 and 8.4.5 holds, as given in Exercise 8.4.4: "Given nonnegative integers g_F, g_G, q, Λ, and λ, satisfying $g_F = q\, g_G + (q-1)(\Lambda - 1)/2$ with $\Lambda \geq \lambda$, $\Lambda = \lambda \pmod 2$ and λ is relatively prime to q, then there is a period q eqivatiant surface of geus g_F with quotient of genus g_G. F should also intersect the z-axis Λ times, and its boundary link the z-axis λ times."

Note. We can combine Theorems 8.4.3, 8.4.4, and 8.4.5 to give the following.

Corollary 8.4.6. Edmond's Conditions.

If K is a periodic knot of period q, then there are nonnegative integers g_G and Λ such that $g(K) = q\, g_G + (q-1)(\Lambda - 1)/2$. If a periodic representative of K has linking number λ with the z-axis, then $\Lambda \geq \lambda$ and $\Lambda = \lambda \pmod 2$, and λ is relatively prime to q.

Revised: 4/9/2021