Section 8.5. Applications of the Murasugi and Edmonds Conditions

Note. In this section we present three corollaries to the Murasugi Conditions (Theorem 8.3.2) and the Edmond's Conditions (Corollary 8.4.6). Each involves specific values of the genus and period of the knot. We give examples to illustrate each. We start with period 3 knots.

Corollary 8.5.7. If a genus 1 knot K has period 3, then its Alexander polynomial satisfies $A_K(t) = \pm t^i(t^2 + 2t + 1) \pmod 3$.

Note. The $(1, -3, 5)$-pretzel knot 7_2 is a genus 1 knot. Its Alexander polynomial is $A_K(t) = -3t^2 + 7t - 3$ by Appendix 2. By Corollary 8.5.7, if 7_2 has period 3 then $A_K(t) = \pm t^i(t^2 + 2t + 1) \pmod 3$. So 7_2 cannot have period 3, since $A_K(t) = t \pmod 3$.

7_2

Corollary 8.5.8. If a genus 2 knot K has period 3, then its Alexander polynomial satisfies $A_K(t) = \pm t^i \pmod 3$.

Note. The knot K in Figure 8.15 has Alexander polynomial $3t^4-7t^3+7t^2-7t+3$. By Theorem 6.2.1, the degree of the Alexander polynomial of a genus g knot is at most $2g$, so the knot of Figure 8.15 must be at least 2. Livingston claims that the Seifert Algorithm of Theorem 4.3.7, when applied to knot K, produces a genus 2 Seifert surface, so that K has genus exactly 2. Since the Alexander polynomial $3t^4 - 7t^3 + 7t^2 - 7t + 3$ is not of the form $\pm t^i$ (mod 3), then by Corollary 8.5.8, K cannot have period 3. This example illustrates the power of the corollaries of this section, since neither the Murasugi Conditions (Theorem 8.3.2) nor Edmonds Conditions (Corollary 8.4.6) apply to conclude that the period of K is not 3.

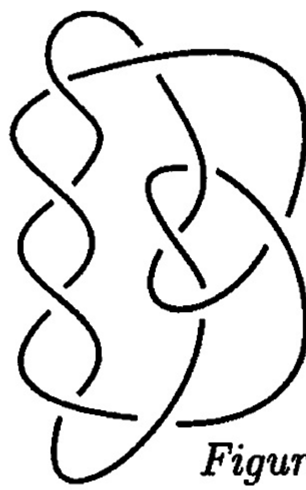

Figure 8.15

Corollary 8.5.9. If a nontrivial knot K is of period 5 and $g(K) \leq 3$, then the Alexander polynomial of K satisfies $A_K(t) = \pm t^i(t^4 - t^3 + t^2 - t + 1)$ (mod 5), and genus$(K) = 2$.

Note. The knot K in Figure 8.16 has Alexander polynomial $5t^4 - 15t^3 + 21t^2 - 15t + 5$. Livingston claims that the Seifert Algorithm of Theorem 4.3.7, when applied to knot K, produces a genus 2 Seifert surface, so that K has genus at most 2 (and hence $g(K) \leq 3$). So *if* K has period 5, then by Corollary 8.5.9 we would have $A_K(t) = \pm t^i(t^4 - t^3 + t^2 - t + 1) \pmod 5$, but $5t^4 - 15t^3 + 21t^2 - 15t + 5 = t^2 \pmod 5$ so K cannot have period 5. Notice that neither the Murasugi Conditions (Theorem 8.3.2) nor Edmonds Conditions (Corollary 8.4.6) apply to conclude that the period of K is not 5.

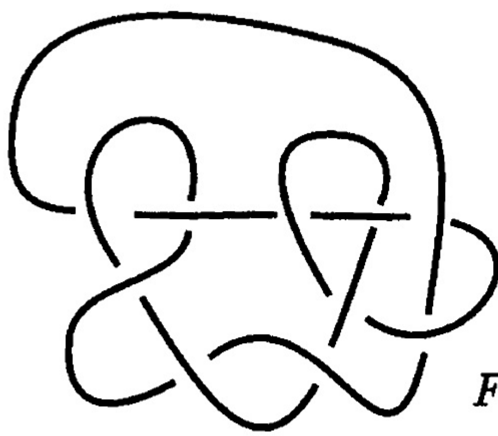

Figure 8.16

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