Section 9.1. Defining Knots in Higher Dimensions

Note. In $\mathbb{R}^3$, to avoid wild knots we had two approaches: (1) defining a knot using smooth (i.e., differentiable) functions on an interval, and (2) defining knots using polygons. In $\mathbb{R}^3$, these yield the same results (namely, the same equivalence classes of knots). In higher dimensions, though, things are more complicated. Livingston states: "For instance, it is true that every smooth knot can be closely approximated by a polygonal knot, and that if two different approximations are chosen, the two polygonal knots are equivalent. However, when two inequivalent smooth knots are approximated by polygonal knots, it is possible that the resulting knots may be polygonally equivalent." See page 181 (Livingston also states that these claims lie "at the foundations of topology and [are] beyond the scope of this chapter"). In higher dimensions (well, higher than 3) we avoid polygonal knots and consider instead "smooth knots."

Definition. The *k-sphere* S^k is the set of unit vectors in $\mathbb{R}^{k+1}$:

$$S^k = \{(x_1, x_2, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1^2 + x_2^2 + \cdots + x_{k+1}^1 = 1\}.$$

Note. We take this opportunity to briefly complain about he distinction between vectors in $\mathbb{R}^n$ and points in $\mathbb{R}^n$. Vectors and points are very different objects. We can add vectors and multiply them by scalars, but they do not have a position. Points have a position, but we cannot add points or multiply them by scalars! Now there is a *natural relationship* between vectors and points (related to putting the vectors in "standard position"); see my online notes for Linear Algebra (MATH 2010) on 1.1. Vectors in Euclidean Spaces, notice the comments on the "geometric interpretation" on page 3 of these notes. The interpretation of all unit vectors in $\mathbb{R}^k$ as a sphere is dependent on this geometric interpretation of vectors and points. It is best the think of S^k as a "manifold" which we will smoothly map into n-space; the "smooth map" will require that we deal with a differentiable structure on S^k and mappings into $\mathbb{R}^n$.

Note. A discussion of surfaces as 2-manifolds can be found in my online notes for Differential Geometry (MATH 5310) on 1.9. Manifolds. A more detailed discussion on smooth k-manifolds can be found in some supplemental notes to the Differential Geometry class on VII.2. Manifolds. Here, we only need to consider the manifold S^k and we approach smoothness only briefly.

Definition. A *smooth knotted k-sphere* in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form $F(S^k)$ where F is a one-to-one differentiable function from S^k to $\mathbb{R}^n$ with everywhere nonsingular derivative.

Note. To say that function $F : S^k \to \mathbb{R}^n$ is differentiable is to say that there is a function (the derivative of F) that assigns each point $p \in S^k$ to a linear map, $D_p(F)$, from the set of tangent vectors to S^k at p into $\mathbb{R}^n$. To say that F has a nonsingular derivative is to say that the image of S^k in $\mathbb{R}^n$ is also a k-dimensional manifold. These ideas are addressed in more detail in my supplemental notes to Differential Geometry VII.2. Manifolds (see Definition VII.2.02).

Definition. Let K_0 and K_1 be smooth k-knots in $\mathbb{R}^n$. These knots are *smoothly equivalent* if there is a family of differentiable functions, F_t where $0 \le t \le 1$, from S^k to $\mathbb{R}^n$ such that:

(1) for all t, F_t is one-to-one with everywhere nonsingular derivative,

(2) $F_0(S^K) = K_0$ and $F_1(S^k) = K_1$, and

(3) the function G from $S^k \times [0,1]$ to $\mathbb{R}^n$ defined by $G(p,t) = F_t(p)$ is differentiable.

Note. Informally, two k-knots in $\mathbb{R}^n$ are equivalent if one can be smoothly transformed into the other through a sequence of smooth knots. We "start" at knot K_0 when $t = 0$ and "end" with knot K_1 when $t = 1$. The nonsingular condition insures that no wild knots arise in the transforming process.

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