Section 9.3. Three-Dimensional Cross Sections of a 4-Dimensional Knot

Note. In this section we consider cross sections of 2-knots in 4-dimensions, and give an example of a coloring of a 2-knot.

Note. In the previous section we considered 2-dimensional cross sections of 3-dimensional objects by letting a plane intersect the object and pass through different parts of the object as a function of time. We can do a similar thing for 4-dimensional objects. Define $H_\tau = \{(x, y, z, t) \in \mathbb{R}^4 \mid t = \tau\}$. Then we can describe a 2-knot K in 4-space by considering a sequence of 3-dimensional cross-sections of the form $H_t \cap K$. For example, if we consider the 2-unknot then we would see the 3-dimensional cross sections start with a point that grows into a circle and then shrinks back down to a point, similar to what we saw in Figure 9.2 (only here the cross sections are in 3-dimensions and in Figure 9.2 the cross sections are in 2-dimensions).

Note. A more complicated example is given in Figure 9.8 (only half of the cross sections are shown, the other half being the cross sections given in Figure 9.8, but in the opposite order).

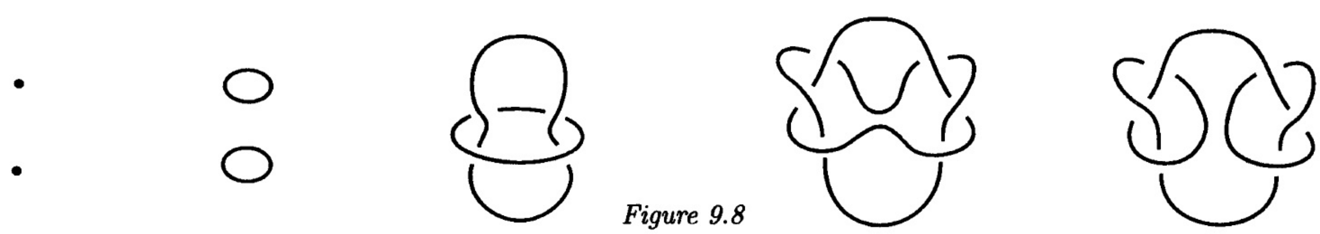


Figure 9.8

This represents the *spun trefoil* 2-knot. It is originally due to Emil Artin (see "Zur isotopic zweidimensionaler Flächen im $\mathbb{R}^4$ [Isotopic Two-dimensional Surfaces in $\mathbb{R}^4$]," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, **4**(1-2), 174-177 (1926)). Livingston claims that it is quite difficult to prove that the spun trefoil is nontrivial and requires a "careful study" of it fundamental group. He also claims "there is no simple generalization of Reidemeisiter moves in dimension 4. (See page 188).

Note. A family of 2-knots was described by E. C. Zeeman in "Twisting Spun Knots," *Transactions of the American Mathematical Society*, **115**, 471–495; a copy is available online on the AMS.org website. Consider the sequence given in Figure 9.8 followed by Figure 9.9 (then followed by Figure 9.8 reversed):

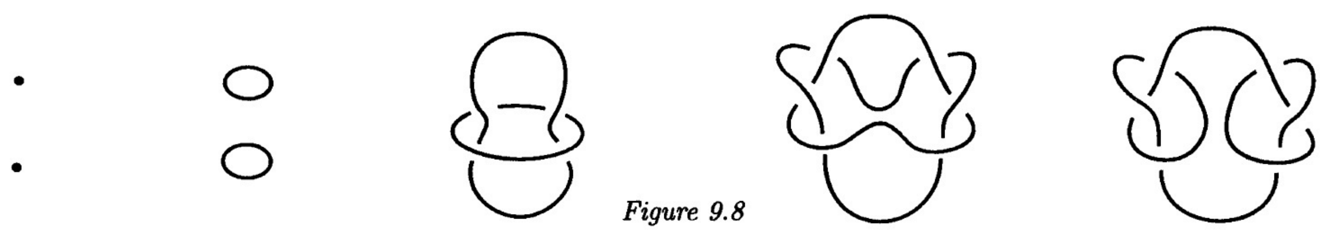


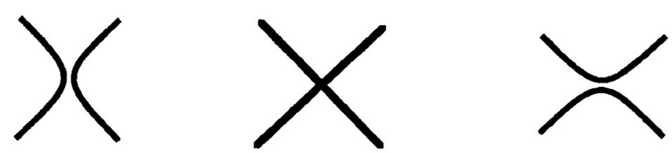
Figure 9.8

Figure 9.9

In Figure 9.9, the trefoil on the right is twisted about its axis k times in Figure 9.9. Notice that with this movement, we are reflecting changes in the 2-knot surface in 4-space. The cross sections are then completed by repeating Figure 9.8 in the reverse order. This is a description of the *k twist spin trefoil*. One of the results in Zeeman's paper if that the 1-twist spin of the trefoil (or the 1-twist of any knot) is unknotted (see his Corollary 2 in Section 6).

Note/Definition. A property of 2-knots in $\mathbb{R}^4$ is that they can be "slightly deformed" (presumably this is an "for every $\varepsilon > 0$ idea) so that all but a finite number of cross-sections are either links or empty. The finite number of cross-sections which are exceptions to this correspond to transitions where components appear or disappear, or components band together of split apart. These second types of exceptions are called *band moves*. Examples of the first type of transitions are given in Figure 9.10 (left) and of the second type are given (band moves) in Figure 9.10 (right).

Figure 9.10



Note. Recall from Section 3.2. Colorings that a knot diagram is colorable if each arc can be drawn using one of three colors in such a way that (1) at least two of the colors are used, and (2) at any crossing at which at least two colors appear, all three colors appear. In Theorem 3.2.2 we saw that, for a given knot or link, if one knot/link diagram is colorable then every knot/link diagram is colorable (the proof was based on Reidemeister moves).

Definition. A 2-knot is *colorable* if every cross-section can be colored so that "nearby" cross-sections are colored in a consistent manner, and at least two colors appear. That is, when new components appear they can receive any color, but when components join together the colorings must be the same at the point where they meet. A (nontrivial) coloring of the spun trefoil is given in Figure 9.11 (though some of the intermediary steps are omitted).

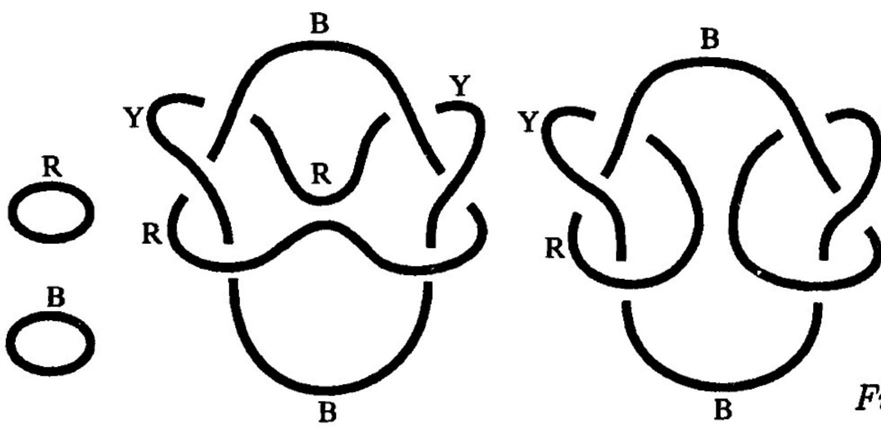


Figure 9.11