Section 9.4. Slice Knots

Note. In this section we define a classical knot as slice if it is a cross section of a 2-knot in $\mathbb{R}^4$. We consider such knots as intersection of other hyperplanes of 4-space. We define a ribbon knot and prove that every ribbon not is slice. We also conjecture that every slice not is a ribbon knot. We also describe the Alexander polynomial of a slice knot.

Definition. A "classical" knot in $\mathbb{R}^3$ that is a cross-section (or *slice*) of 2-knots in $\mathbb{R}^4$ is a *slice knot*.

Note/Definition. We can identity 3-space $\mathbb{R}^3$ with the hyperplane of 4-space $H_0 = \mathbb{R}^4_+ = \{(x, y, z, t) \mid t \geq 0\}$. A (classical) knot in $\mathbb{R}^3$ is slice if it is the boundary of a *smooth disk* in $H_0 = \mathbb{R}^4_+$ and the disk it bounds is called its *slice disk*. Figure 9.16 gives a humble example of a slice disk.

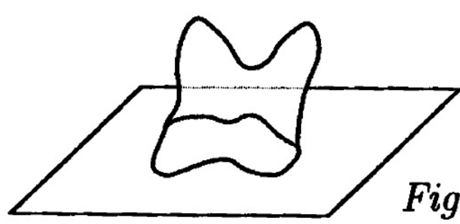


Figure 9.16

Note. In Exercise 9.3.5 it was to be shown that every connected sum of a knot and its mirror image is slice; more specifically, $K\#K^{rm}$ is slice).

Definition. A knot is a *ribbon knot* if it bounds a disk with self intersections only of the type given in Figure 9.17. Such a disk is a *ribbon disk*.

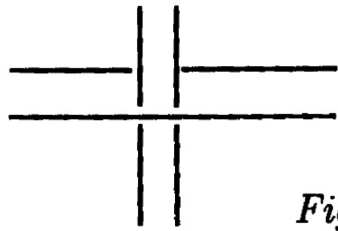
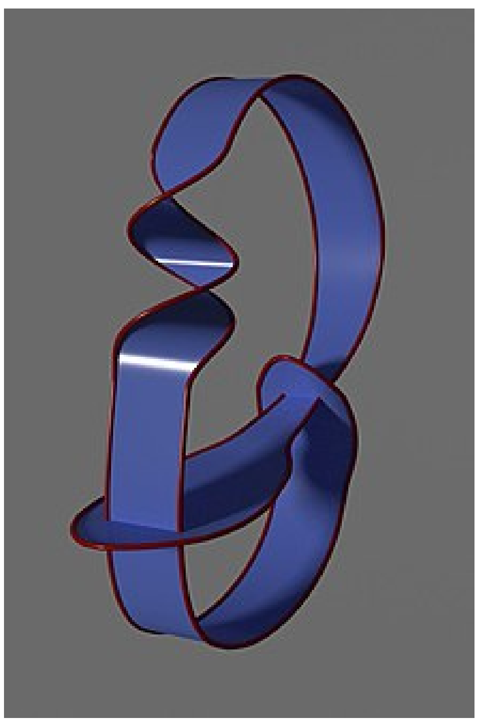


Figure 9.17

Note. The Wikipedia Ribbon knot webpage clarifies this idea. It describes a ribbon knot as: "...a ribbon knot is a knot that bounds a self-intersecting disk with only ribbon singularities. Intuitively, this kind of singularity can be formed by cutting a slit in the disk and passing another part of the disk through the slit." This is illustrated with the following image (also from the Wikipedia page):



Theorem 9.4.1. Every ribbon knot is slice.

Note. Livingston gives an argument based on Figure 9.18. He claims that the ribbon intersections can be "pinched" to divide the knot into several components. The collection of components forms an unlink and each can be shrunk to a point, and so these are a cross section of 2-knots.

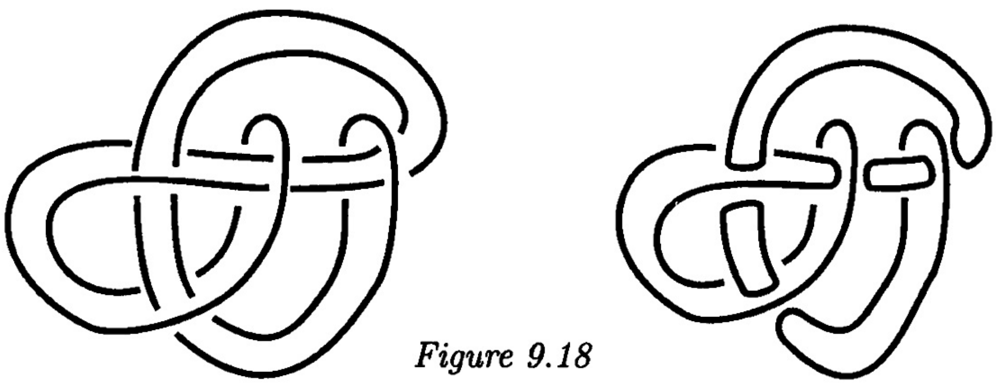


Figure 9.18

Note. I was conjectured by Ralph Fox that the converse of Theorem 9.4.1 holds. This was stated as Problem 25 in R. H. Fox, "Some Problems in Knot Theory," *Topology of 3-manifolds and Related Topics* (Proceedings of The Univiveristy of Georgia Institute, 1961), Englewood Cliffs, N.J.: Prentice-Hall, pages 168176 (1962); available online on the webpage of Benjamin Matthias Ruppik (accessed 5/1/2021). Specifically, it states:

Conjecture. Every slice knot is ribbon.

Note. The following result puts constraints on the Seifert matrices of a slice knot. As shown in the corollaries following it, constraints on the Alexander polynomial and signature follow. The proof of the next theorem "is beyond what can be presented here," but a proof can be found in Raymond Lickorish's *An Introduction Knot Theory*, Graduate Texts in Mathematics 175, (NY: Springer-Verlag, 1997) (see Proposition 8.17 in Chapter 8. The Conway Polynomial, Signatures and Slice Knots).

Theorem 9.4.2. If a knot K is slice and V is any Seifert matrix arising from a Seifert surface of genus g, then there is an invertible (determinant 1) integer matrix M such that MVM^t is of the form $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$ where B, C, and D are $g \times g$ matrices with $B - C = \pm I_g$, where I_g is the $g \times g$ identity matrix.

Corollary 9.4.3. The Alexander polynomial of a slice knot can be factored as $\pm t^k f(t) f(-t)$ for some integer polynomial f and integer k.

Note. The Alexander polynomial for the trefoil knot, $t^2 - t + 1$, is irreducible since it's roots are $t = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \dfrac{1 \pm \sqrt{-3}}{2}$. So by Corollary 9.4.3, it is not a slice knot.

Corollary 9.4.4. If a knot is slice then its signature (and all its ω-signatures) are 0.

Definition. If a knot has a Seifert form which is similar to a matrix of the form $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$, then it is *algebraically slice.*

Note. We have from Theorem 9.4.2 that all slice knots are algebraically slice. Livingston claims that a "high-dimensional" knot is slice if and only if it is algebraically slice. He references

1. Michel Kervaire, *Knot Cobordism in Codimension Two*, Lecture Notes in Mathematics, Springer Verlag, Berlin-Heidelberg-New York (1971).

2. J. Levine, "Knot Cobordism Groups in Codimension Two," *Commentarii Mathematici Helvetici*, **45**, 229–244 (1969)

Kervaire's paper is available through Andrew Ranicki's Homepage. Levine's papers is also available through Andrew Ranicki's Homepage (both accessed 5/2/2021).

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