Section 9.5. The Knot Concordance Group

Note. In this section we define an equivalence relation called "concordance" on the set of classical (that is, knots in 3 dimensions). This relation is based on slice knots (and that's why it appears at this point in the book). The operation of connected sum on the equivalence classes of knots under concordance yields a group called the "concordance group." We justify these statements and explore the structure of the concordance group.

Definition. Knots K and J are *concordant* if $K\#J^{rm}$ is slice.

Note. We now claim that concordance is, in fact, an equivalence relation.

Theorem 9.5.5. Concordance forms an equivalence relation on the set of knots.

Note. By Exercise 9.3.5, we have that for any knot K, the knot $K\#K^{rm}$ is slice and so concordance is reflexive. Livingston claims that symmetry is "automatically satisfied." Transitivity is to be demonstrated in Exercise 9.5.1.

Note. Before we present the concordance group, we need a preliminary lemma.

Lemma 9.5.6. If K_1 is concordant to K_2 and J_1 is concordant to J_2, then $K_1\#J_1$ is concordant to $K_2\#J_2$.

Note. We now use Lemma 9.5.6 to show that the connected sum operation yields a well-defined operation of the equivalence classes under the equivalence relation of concordance. That is, the binary operation in the concordance group can be evaluated using representatives from the equivalence classes. For more on this idea of a well-defined binary operation, see my online notes for Introduction to Modern Algebra (MATH 4127/5127) on Section III.14. Factor Groups; see Theorem 14.4.

Theorem 9.5.7. With respect to the operation induced by connected sum, the set of concordance classes of knots forms an abelian group, denoted C_1^3, and called the *concordance group*.

Note. A survey of results from several years ago is Chapter 7. A Survey of Classical Knot Concordance by Charles Livingston (accessed 5/2/2021). This appears in *Handbook of Knot Theory*, edited by William Menasco and Morwen Thistlethwaite, Elsevier Science, pages 319–347 (2005). Livingston in the text book states that "little more is known" than the following properties (see page 203). Remember, the text book is copyright 1996.

Note. The following four "facts" about the concordance group C_1^3 are the following (along with some justification).

1. The concordance group is countable.

Every knot (in $\mathbb{R}^3$ and given as a polygonal knot) can be deformed so that its (finite number of) vertices have rational coordinates. So the set of (classical) knots is countable and hence the number of concordance classes of knots is countable. Therefore the concordance group (having the concordance classes as elements) is countable.

2. The function that maps knot K to $\sigma(K)/2$, where $\sigma(K)$ is the signature of a knot (see Section 6.3. Signature of a Knot, and Other S-equivalent Invariants), is a homomorphism from the concordance group onto $\mathbb{Z}$, and hence the concordance group is infinite.

By Exercise 6.3.6, the signature of a connected sum is the sum of the constituent signatures (that is, $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$), so the mapping $K \mapsto \sigma(K)/2$ is a homomorphism (by Exercise 6.3.4, $\sigma(K)$ is always even). By Example 6.3.A, the left-handed trefoil knot has signature 2 and the right-handed trefoil knot has signature -2, so the mapping is surjective (onto) $\mathbb{Z}$ (since we can consider multiple connected sums of the left-handed or right-handed trefoil to get a knot with signature $2n$ for any $n \in \mathbb{Z}$).

3. There are elements of order 2 in the concordance group.

The figure-8 knot $K = 4_1$ has an irreducible Alexander polynomial, $t^2 - 3t + 1$ (see Appendix 2) and so is not slice by Corollary 9.4.3. The figure-knot is negative amphicheiral (that is, $K = K^{rm}$; see Note 8.1.A), so $K \# K = K \# K^{rm}$ is slice and, since the identity in the concordance group is the set of all knots concordant with the unknot (see the proof of Theorem 9.5.7) so that the figure-8 knot is of order 2.

4. There is a homomorphism from the concordance group that maps onto $\mathbb{Z}^\infty$.

Livingston mentions using an infinite collection of homomorphisms to $\mathbb{Z}$ defined using ω-signatures, to define the claimed homomorphism. He states that Levine found knots which establish surjectivity.

Revised: 5/4/2021