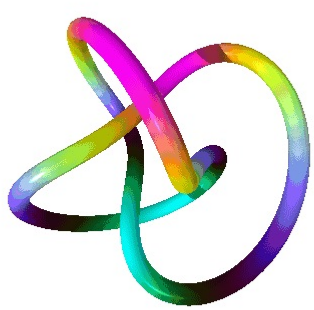
Chapter 9. High-Dimensional Knot Theory

Note. We could naively extend the idea of a knot in 3-space to the idea of a knot in 4-space by simply considering such a knot as a simple closed polygonal curve in 4-space (in analogy to the definition given in Section 2.2. The Definition of a Knot). The result is called a 1-knot in 4-space. However, as Livingston states: "It turns out though that there is really no interesting theory of such 1-knots; all such knots in 4-space are equivalent." (See page 179.) That is, all 1-knots in 4-space are equivalent to the unknot! So "you can't tie your shoes in four dimensions." The idea behind this claim as that any crossings can be reversed in 4-space (that is, "overcrossings" can be changed to "undercrossings"). At the crossing, simply take one strand and lift it in the direction of the 4th dimension (i.e., lift it "hyper-up"; for an elementary explanation of this idea, see my online notes for Foundations and Structure of Mathematics 1 [MATH 5025] on THE FOURTH DIMENSION (AND MORE!)), move it over the other strand, and lay it back down from the 4th dimension (i.e., set it "hyper-down"). A nice little animated image of this is given on a webpage created by Oliver Knill of Harvard University on Unknotting a Knot in 4D (accessed 4/14/2021) where color is used to represent the 4th dimension:



A still image from the Unknotting a Knot in 4D website

Note. In light of this observation, we need to modify the idea of a knot in higher dimensions. We do so in Section 9.1 by both increasing the dimension of the space from 3 to n and increasing the dimension of the knot from 1 to k. Whereas we could consider a knot in 3-space as a homeomorphic image of a circle, we will deal with knots in n-space as ("smooth") images of k-dimensional spheres.

Note. In Sections 9.2 and 9.3, we deal with properties and visualization of knots in higher dimensions. In Section 9.4 we consider the *sliceness* of knots in 3-space, a property based on knotted 2-spheres in 4-space. In Section 9.5 we consider a group (the *knot concordance group*) on a set of classes of knots.

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