Chapter 2. Motion Along a Straight Line

Section 2.1. Position, Displacement, and Average Velocity

Note. The motion of objects is a central part of physics. In this chapter we consider motion along a line, also called *one-dimensional motion* or *rectilinear motion*. In Chapter 3 we review vectors and in Chapter 4 we consider motion in two and three dimensions. Halliday and Resnick define the classification and comparison of motions as *kinematics*. In an engineering program, the study of motion is covered in a class on "Dynamics." I have a small amount of information online for this class on my Dynamics (MATH 2620) webpage (this class was formerly co-listed as MATH 2620, PHYS 2620, and ENTC 2620 ("ENTC" was the Department of Engineering Technology). Today, as part of the TTU-ETSU Bachelor of Science in Engineering program, this class is "Dynamics" (ME 2330).

Note. In addition to restricting our attention in this chapter to motion along a line, we also restrict our attention to a consideration of speed and acceleration but we do not directly consider forces until Chapter 5. We assume that the moving object is a point-like object that we refer to as a *particle*.

Note. Since we consider motion along a line, we introduce a coordinate line or axis. The *positive direction* of the axis is in the direction of increasing numbers (or "coordinates"), and the *negative direction* of the axis is in the direction of decreasing coordinates. We may consider the coordinate axis as horizontal, but several applications will make more sense if we consider the axis as vertical with the positive direction as upward. See Figure 2.1.

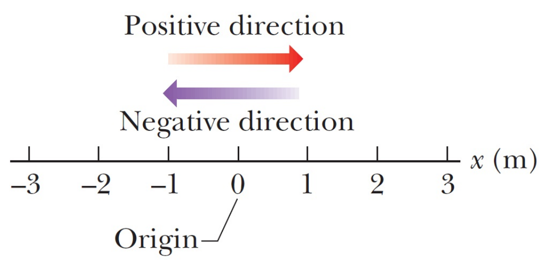


Figure 2-1 Position is determined on an axis that is marked in units of length (here meters) and that extends indefinitely in opposite directions. The axis name, here x, is always on the positive side of the origin.

We must keep up with units when we consider locations of particles on the coordinate axis, so that the coordinates themselves carry these units. In Figure 2-1, the units are meters. A change from position x_1 to position x_2 gives a *displacement* of $\Delta x = x_2 - x_1$. Notice that a positive displacement is associated with movement in the positive direction (and similarly for negative displacement). The *magnitude* of a displacement is the absolute value of the displacement, $|\Delta x| = |x_2 - x_1|$ (so displacement also includes the units of distance). Displacement is a *vector quantity*, which means that it comes both with a magnitude and a direction (more on this in Chapter 3). For motion along a line, the direction is indicated by the sign (positive or negative) of the displacement.

Definition. If a particle in motion has coordinate x_1 at time t_1 and coordinate x_2 at time t_2, then the *average velocity* of the particle from time t_1 to time t_2 is

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}. \tag{2-2}$$

Notice that the units of average velocity are units of distance divided by units of time (such as meters per second).

Note. You encountered the idea of average speed early in Calculus 1 (MATH 1910). See my online notes for Calculus 1 on Section 2.1. Rates of Change and Tangents to Curves (where the "speed" is allowed to be negative, unlike here). In Calculus 1, this average rate of change was used to inspire the use of limits to consider instantaneous rates of change. The average velocity represents the slope of the secant line to the graph of position as a function of time over the time interval from t_1 to t_2.

Definition. The *average speed* of a moving particle over time interval $[t_1, t_2]$ is the total distance covered (whether in the positive or negative direction) divided by the time $\Delta t = t_2 - t_1$:

$$s_{\text{avg}} = \frac{\text{total distance}}{\Delta t}. \qquad \text{(2-3)}$$

Problem 2.6. The 1992 world speed record for a bicycle (human-powered vehicle) was set by Chris Huber. His time through the measured 200 m stretch was a sizzling 6.509 s. In 2001, Sam Whittingham beat Hubers record by 19.0 km/h. What was Whittingham's time through the 200 m?

Section 2.2. Instantaneous Velocity and Speed

Definition. Consider a particle moving along a line with x-coordinate a differentiable function of t given by $x(t)$. The *instantaneous velocity* (or simply *velocity*) as a function of time is

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}. \qquad \text{(2-4)}$$

The *speed* (as opposed to average speed) is the magnitude of velocity, $|v(t)|$.

Problem 2.16. The position function $x(t)$ of a particle moving along an x axis is $x = 4.0 - 6.0t^2$, with x in meters and t in seconds. **(a)** At what time and **(b)** where does the particle (momentarily) stop? At what **(c)** negative time and **(d)** positive time does the particle pass through the origin? **(e)** Graph x versus t for the range -5 s to $+5$ s. **(f)** To shift the curve rightward on the graph, should we replace the term 't' with $t + 20$ or $t - 20$ in $x(t)$ (this differs from the text book's statement)? **(g)** Does that inclusion increase or decrease the value of x at which the particle momentarily stops?

Section 2.3. Acceleration

Definition. Consider a particle moving along a line with x-coordinate. The particle *accelerates* if it's velocity changes. The *average acceleration*, a_{avg}, over a time interval $[t_1, t_2]$ is

$$a_{\text{avg}} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}, \qquad \text{(2-7)}$$

where the particle has velocity v_1 at time t_1 and velocity t_2 at time t_2. If the velocity is a differentiable function of time $v(t)$ (so that position $x(t)$ is a twice differentiable function of time) then the *instantaneous acceleration* (or simply *acceleration*) is

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \qquad \text{(2-8, 2-9)}$$

Note. The units of acceleration are of the form length per time^2; for example, we will commonly use m/s^2. Sometimes the unit of "g" is used to measure acceleration, where 1 g = 9.8 m/s^2 (this will be motivated when we discuss force in Chapters 5 and 6). Acceleration has both magnitude and direction where, as with velocity, the direction for motion along a line is given by a positive or negative sign. We feel acceleration as a "pull" (or force). When riding in a car and going around a sharp corner, we feel pulled to one side of the car due to the acceleration which produces a change in velocity (the *direction* of travel is changing). Similarly, if we slam on the brakes we feel pulled towards the front of the car. When going quickly over the top of a steep hill, we can enter a brief amount of time in free-fall in which the pull of gravity is lessened and we feel this as the sensation of falling (in zero gravity, one continually feels as if they are falling).

Problem 2.20. **(a)** If the position of a particle is given by $x = 20t - 5t^3$, where x is in meters and t is in seconds, when, if ever, is the particle's velocity zero? **(b)** When is its acceleration a zero? **(c)** For what time range (positive or negative) is a negative? **(d)** Positive? **(e)** Graph $x(t)$, $v(t)$, and $a(t)$.

Section 2.4. Constant Acceleration

Note. We now consider the special case (which is a common case) of constant acceleration. First we approach this without the use of calculus. In this case, acceleration (as a function of time) equals average acceleration, so with time t as a variable, we have $a(t) = a_{\text{avg}} = \dfrac{v(t) - v_0}{t - 0}$ or (suppressing the variable t)

$a = (v - v_0)/(t - 0)$. This implies (solving for $v(t) - v$):

$$v = v_0 + at \text{ or } v(t) = v_0 + at. \tag{2-11}$$

Similarly, if we compute the average velocity as a function of time we have from Equation (2-2) that over the time interval $[0, t]$:

$$v_{\text{avg}} = \frac{x - x_0}{t - 0} \text{ or } v_{\text{avg}}(t) = \frac{x(t) - x_0}{t - 0},$$

which gives

$$x = x_0 + v_{\text{avg}}t \text{ or } x(t) = x_0 + tv_{\text{avg}}(t). \tag{2-12}$$

We now can compute position $x(t)$ in terms of initial position x_0 and initial velocity v_0 in the case of constant acceleration. Notice that if acceleration is constant then the graph of velocity in terms of time (that is, the graph of $y = v(t)$) is a line of slope equal to the acceleration (since is the rate of change of position). Then $v_{\text{avg}} = (v_0 + v(t))/2$ on the interval $[0, t]$. We now have:

$$\begin{aligned}
x(t) &= x_0 + tv_{\text{avg}}(t) \text{ from Equation (2-12)} \\
&= x_0 + t\frac{v_0 + v(t)}{2} \text{ since } v_{\text{avg}} = (v_0 + v(t))/2 \text{ on } [0, t] \\
&= x_0 + tv_0/2 + tv(t)/2 = x_0 + tv_0/2 + t(v_0 + ta)/2 \text{ by Equation (2-11)} \\
&= x_0 + v_0 t + \frac{1}{2}at^2.
\end{aligned}$$

That is,

$$x - x_0 = v_0 t + \frac{1}{2}at^2 \text{ or } x(t) - x_0 = v_0 t + \frac{1}{2}at^2. \tag{2-15}$$

Halliday and Resnick refer to Equations (2-11) and (2-15) as the *basic equations for constant acceleration.* We now create several other equations by eliminating certain variables using Equations (2-11) and (2-15). Solving for t in (2-11) gives $t = (v - v_0)/a$, which upon substituting into (2-15) yields $x(t) - x_0 = v_0((v(t) - v_0)/a) + \frac{1}{2}a((v(t) - v_0)/a)^2$ or (multiplying both sides by $2a$) $2a(x(t) - x_0) = 2v_0v(t) - 2v_0^2 + (v(t))^2 - 2v(t)v_0 + v_0^2$ or (simplifying and solving for $v(t)$) $(v(t))^2 = v_0^2 + 2a(x(t) - x_0)$. That is,

$$v^2 = v_0^2 + 2a(x(t) - x_0) \text{ or } (v(t))^2 = v_0^2 + 2a(x(t) - x_0). \tag{2-16}$$

Solving for a in (2-11) gives $a = (v(t) - v_0)/t$, which upon substituting into (2-15) yields $x(t) - x_0 = v_0t + \frac{1}{2}(v(t) - v_0)/t)t^2$ or (simplifying) $x(t) - x_0 = \frac{1}{2}((v_0 + v(t))t$. That is,

$$x - x_0 = \frac{1}{2}(v_0 - v)t \text{ or } x(t) - x_0 = \frac{1}{2}((v_0 + v(t))t. \tag{2-17}$$

Solving for v_0 in (2-11) gives $v_0 = v(t) - at$, which upon substituting into (2-15) yields $x(t) - x_0 = (v(t) - at)t + \frac{1}{2}at^2$ or (simplifying) $x(t) - x_0 = tv(t) - \frac{1}{2}at^2$. That is,

$$x - x_0 = vy - \frac{1}{2}at^2 \text{ or } x(t) - x_0 = tv(t) - \frac{1}{2}at^2. \tag{2-18}$$

In each of the above equations, four out of the five of the quantities $x - x_0$, v, t, a, and v_0 are present. A summary is given in Table 2-1:

Table 2-1 **Equations for Motion with Constant Acceleration**[a]

Equation Number	Equation	Missing Quantity
2-11	$v = v_0 + at$	$x - x_0$
2-15	$x - x_0 = v_0t + \frac{1}{2}at^2$	v
2-16	$v^2 = v_0^2 + 2a(x - x_0)$	t
2-17	$x - x_0 = \frac{1}{2}(v_0 + v)t$	a
2-18	$x - x_0 = vt - \frac{1}{2}at^2$	v_0

[a]Make sure that the acceleration is indeed constant before using the equations in this table.

Problem 2.28. On a dry road, a car with good tires may be able to brake with a constant deceleration of 4.92 m/s^2. **(a)** How long does such a car, initially traveling at 24.6 m/s, take to stop? **(b)** How far does it travel in this time? **(c)** Graph x versus t and v versus t for the deceleration.

Note. Now, we take advantage of the power of calculus to give alternate (and more intuitive) derivations of the basic equations (2-11) and (2-15). Halliday and Resnick take some liberties with differentials (as is common in applied courses), so we give derivations that give full respect to the Chain Rule and the Substitution Rule (see my online Calculus 1 on Section 5.5. Indefinite Integrals and the Substitution Method; notice Theorem 5.6, The Substitution Rule). We also make a clear distinction between an antiderivative and an indefinite integral (see my Calculus 1 notes on Section 5.3. The Definite Integral for details). By Equation 2-8 we have $dv/dt = a$, so we have the indefinite integral

$$\int \frac{dv}{dt}\, dt = \int a\, dt \text{ or } v(t) = at + C = \{at + k \mid k \in \mathbb{R}\}.$$

Here, $+C$ represents an "arbitrary constant"; technically, an indefinite integral is the set of all antiderivatives of the integrand. We are looking for a particular antiderivative of dv/dt that satisfies the initial condition that when $t = 0$ we have $v(0) = v_0$. So we choose a particular constant of integration k, such that $v(t) = at + k$ and $v(0) = v_0$. This requires that $v(0) = a(0) + k = v_0$, so that we need $k = v_0$. Hence, $v(t) = at + v_0$ which is equation (2-11). Since $dx/dt = v(t)$ by Equation (2-4), then (by equation (2-15)):

$$\int \frac{dx}{dt}\, dt = \int v(t)\, dt = \int at + v_0\, dt = \frac{1}{2}at^2 + v_0 t + C$$

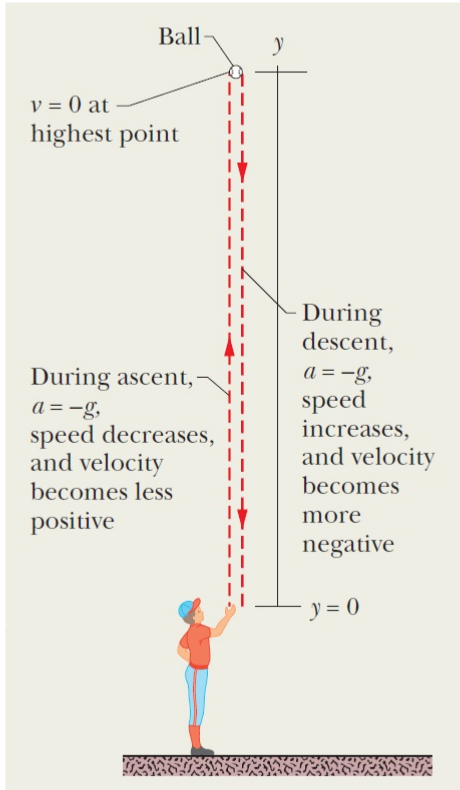
$$\text{or } x(t) = \frac{1}{2}at^2 + v_0 t + C = \left\{ \frac{1}{2}at^2 + v_0 t + k \mid k \in \mathbb{R} \right\}.$$

We are looking for a particular antiderivative of $x(t)$ that satisfies the initial condition that when $t = 0$ we have $x(0) = x_0$. So we choose a particular constant of integration k, such that $x(t) = \frac{1}{2}at^2 + v_0 t + k$ and $x(0) = x_0$. This requires that $x(0) = \frac{1}{2}a(0)^2 + v_0(0) + k = x_0$, so that we need $k = x_0$. Hence, $x(t) = \frac{1}{2}at^2 + v_0 t + x_0$ which is equation (2-15).

Section 2.5. Free-Fall Acceleration

Note. An intuitive example of motion along a line, is the idea of throwing an object straight upward. We place the coordinate axis vertically with the positive direction upward; we therefore use the coordinate y for the position of the object. We take this motion as occurring on the surface of the Earth so that the acceleration has magnitude g in the negative (downward) direction. So we have $a = -g = -9.8$ m/s^2. This is called *free-fall acceleration*. With y_0 as the initial height (the height at which the object is released) and v_0 is its initial velocity, we have the height (that is, the y coordinate) as a function of time as $y = -9.8t^2 + v_0 t + y_0$, where t is measured in s, v_0 is measured in m/s, and y and y_0 are measured in m.

Sample Problem 2.05. A pitcher tosses a baseball up along a y axis with an initial speed of 12 m/s. See Figure 2-13. **(a)** How long does the ball take to reach its maximum height? **(b)** What is the ball's maximum height above its release point? **(c)** How long does the ball take to reach a point 5.0 m above its release point?



Section 2.6. Graphical Integration in Motion Analysis

Note. If acceleration is a Riemann integrable function of time, $a(t)$, (which it is, if $a(t)$ is continuous or, equivalently, if $v(t)$ is continuously differentiable; see my online notes for Calculus 1 on Section 5.4. The Fundamental Theorem of Calculus; notice Theorem 5.2(b)), then by the Fundamental Theorem of Calculus

$$\int_{t_0}^{t_1} a(t)\,dt = \int_{t_0}^{t_1} \frac{dv}{dt}\,dt = v(t_1) - v(t_0) = v_1 - v_0.$$

Notice that the integrand in the left-most integral has units of (say) m/s^2, for the acceleration, times s (for dt), or m/s as expected since the value of the definite integral is a velocity. Since definite integrals have an area interpretation (with appropriate care taken when the function is negative; see my online Calculus 1 notes on Section 5.2. Sigma Notation and Limits of Finite Sums; see Figure 5.9), then we have the following:

$$\int_{t_0}^{t_1} a(t)\,dt = \left(\begin{array}{c}\text{area between acceleration curve}\\ \text{and time axis, from } t_0 \text{ to } t_1\end{array}\right). \qquad (2\text{-}28)$$

Note. Similarly, if velocity is a Riemann integrable function of time, $v(t)$ (which it is, if $v(t)$ is continuous or, equivalently, if $x(t)$ is continuously differentiable), then by the Fundamental Theorem of Calculus

$$\int_{t_0}^{t_1} v(t)\,dt = \int_{t_0}^{t_1} \frac{dx}{dt}\,dt = x(t_1) - x(t_0) = x_1 - x_0.$$

Notice that the integrand in the left-most integral has units of (say) m/s, for the velocity, times s (for dt), or m as expected since the value of the definite integral is a position. With the area interpretation of a definite integral (as above), we have the following:

$$\int_{t_0}^{t_1} v(t)\,dt = \left(\begin{array}{c}\text{area between velocity curve}\\ \text{and time axis, from } t_0 \text{ to } t_1\end{array}\right). \qquad (2\text{-}30)$$

See Figure 2-14 for a graphical interpretation of Equations (2-28) and (2-30).

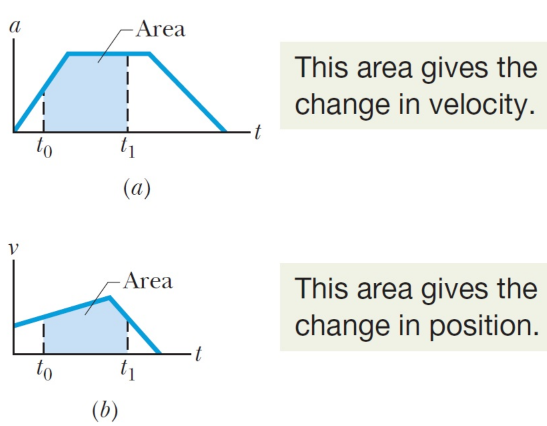


Figure 2-14 The area between a plotted curve and the horizontal time axis, from time t_0 to time t_1, is indicated for (a) a graph of acceleration a versus t and (b) a graph of velocity v versus t.

Problem 2.66. In a forward punch in karate, the fist begins at rest at the waist and is brought rapidly forward until the arm is fully extended. The speed $v(t)$ of the fist is given in Figure 2-37 for someone skilled in karate. The vertical scaling is set by $v_s = 8.0$ m/s. How far has the fist moved at **(a)** time $t = 50$ ms and **(b)** when the speed of the fist is maximum?

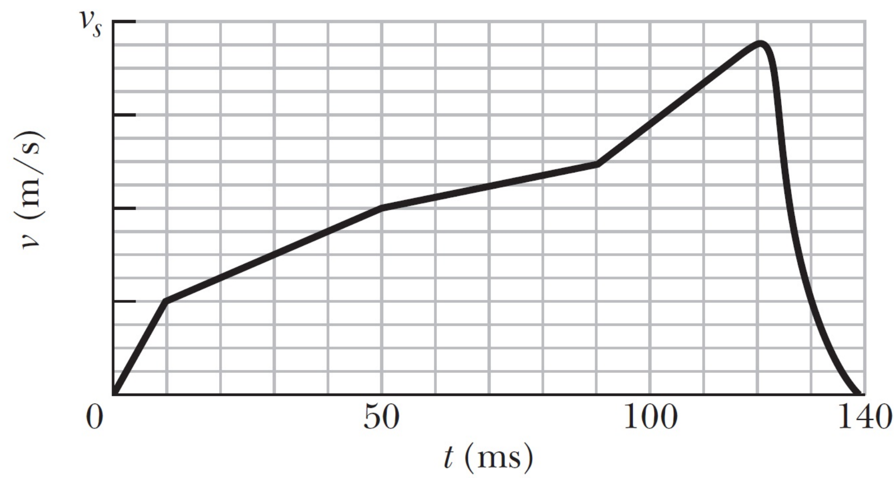


Figure 2-37 Problem 66.

Created: 8/7/2022; Last Revised: 6/27/2023