Chapter 3. Vectors

Note. In your high school experience, you may have heard of a vector described as an entity with both "magnitude and direction." This is also the approach we will take in this chapter. However, this is vague and lacks mathematical rigor. In Linear Algebra (MATH 2010), you cover vectors spaces, vectors, and inner products (or "dot products"). In this setting, a vector is something that satisfies the definition of vector! A vector space can consist of vectors which are functions and scalars which are real numbers (in fact, you will see in upper-level applied math and physics classes that these are some of the most useful applications of vector spaces). It is not at all apparent that a *function* has a magnitude or direction. For more details, see my online notes for Linear Algebra; notice in particular Section 3.1. Vector Spaces. However, in the n-dimensional vector space $\mathbb{R}^n$, it is true that a vector has a magnitude and direction. The magnitude is simply the square root of the sum of the squares of the components of the vector (or, the square root of the inner product of the vector with itself). The concept of "direction" is not formally defined, but it is related to the standard orthonormal basis of $\mathbb{R}^n$. In this course, we are only interested in vectors in $\mathbb{R}^1$ (which we dealt with in Chapter 2. Motion Along a Straight Line), $\mathbb{R}^2$, and $\mathbb{R}^3$ (we cover vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ in Chapter 4. Motion in Two and Three Dimensions). Since these are the only vector spaces we consider, then we are fine in using the "magnitude and direction" approach.

Note. It might be surprising to hear that there are vector spaces in which "direction" is not defined. This is a rather involved result related to the desired properties of a basis. This is addressed in the concepts of a *Schauder basis* versus a *Hamel basis*. An idea of orthogonality is also needed to take full advantage of the "direction" idea. For more on these concepts, see my online notes for Fundamentals of Functional Analysis (MATH 5740), in particular the notes on Section 2.11. Schauder Basis and Supplement. Groups, Fields, and Vector Spaces. It turns out that every vector space has a basis "in the sense of Hamel" (that is, every vector space has a basis where we take the definition of "basis" as given in Linear Algebra); a proof of this is given in "Supplement. Groups, Fields, and Vector Spaces" and it requires something called "Zorn's Lemma" from set theory. The definition of "vector space" only considers the algebraic properties of vectors and their interaction with scalars (in general, the scalars come from an algebraic structure called a "field"; in this class, the field will always be $\mathbb{R}$). However, the definition does not include the existence of a basis. This is *proved* using Zorn's Lemma. Similarly, a norm (which is what we would use to describe the magnitude of a vector) is not part of the definition of a vector space. This is a bit trickier to explore! A discussion of the existence (or nonexistence) of a norm in a general vector space lies in the realm of "metrizable and nonmetrizable topological vector spaces." For our practical physical explorations, none of this affects us because we work in the very concrete vectors spaces of $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

Note. Another common (and useful) collection of vector spaces are those which have complex scalars (as opposed to real scalars). Complex numbers, $\mathbb{C}$, are particularly useful in addressing eigenvalues of matrices. See my online notes for Linear Algebra (MATH 2010) on Section 9.2. Matrices and Vector Spaces with Complex

Scalars. As a final theoretical comment, we observe that your sophomore linear algebra class was basically a study of finite dimensional vector spaces (probably with real scalars) and the linear transformations between them (recall that a linear transformation is represented by a matrix). These have a number of practical applications, including to systems of linear ordinary differential equations; see my online notes for A Second Course in Differential Equations (not an official ETSU class, though these topics are likely covered in ETSU's Introduction to Applied Mathematics [MATH 4027/5027]) on Section 7.5. Homogeneous Linear Systems with Constant Coefficients. But infinite dimensional vector spaces can be quite useful as well. In fact, quantum mechanics is performed in an infinite dimensional vector space (which has both a norm and an inner product) called a *Hilbert space*. A rigorous presentation of Hilbert spaces is given in ETSU's graduate math classes of Real Analysis 2 (MATH 5220; notice Chapter 16) and Fundamentals of Functional Analysis (MATH 5740; notice Chapter 4). For specific applications to quantum mechanics, see my online notes (in preparation as of fall 2022) for Hilbert Spaces and Quantum Mechanics (not an official ETSU class). We now return to Halliday and Resnick's approach to vectors.

Section 3.1. Vectors and Their Components

Note. As described at length in the introduction to Chapter 3 above, the mathematical concept of a vector is much more complicated that an object with "magnitude and direction." Keep this in mind! None-the-less, we follow the text book and present the following definition. This is acceptable in this setting, but beware that it is an oversimplification in other settings!

Definition. A *vector* has magnitude as well as direction. A *vector quantity* is a quantity that has both magnitude and a direction and thus can be represented with a vector. A vector that represents displacement (that is, change of position) is a *displacement vector*. A physical quantity that does not involve direction is a *scalar*.

Note. Examples of vector quantities are velocity and acceleration. In Calculus 3 (MATH 2110), you deal with velocity and acceleration, but not with a "displacement vector." Instead, you introduce a *position vector function* that represents the *position* of a particle at time t (requiring that the position vector be put in "standard position" with its tail at the origin). The first derivative of this position function with respect to time t is the velocity function, the second derivative of the position function with respect to time is the acceleration function, and the third derivative of the position function with respect to time is the "jerk" function. See my online Calculus 3 notes for Chapter 13, "Vector-Valued Functions and Motion in Space," in particular on Section 13.1. Curves in Space and Their Tangents and Section 13.2. Integrals of Vector Functions; Projectile Motion. Examples of scalar quantities are mass, time, length, and temperature (notice that these quantities may be negative, such as in the case of temperature).

Note. We draw vectors as arrows. If a particle changes its position from point A to point B then, as Halliday and Resnick treat the displacement vector, the displacement vector is given by an arrow with its tail at point A (where the particle started) and its head at point B (where the particle ended up). In Figure 3-1(a), we see three displacement vectors. However, since each has the same magnitude and direction they are actually the same vector (remember, *location* is not part of the vector concept, but only magnitude and direction). In Figure 3-1(b), we see how a displacement vector is only based on initial and final position of a particle and it contains no information about the movement of the particle between these positions.

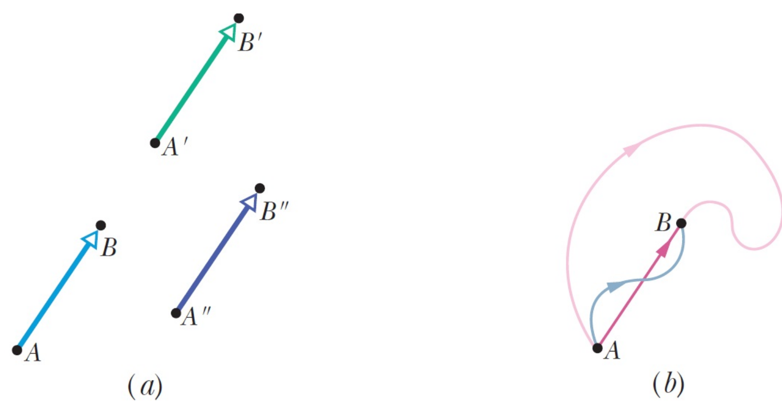

Figure 3-1 (*a*) All three arrows have the same magnitude and direction and thus represent the same displacement. (*b*) All three paths connecting the two points correspond to the same displacement vector.

Note/Definition 3.A. Halliday and Resnick motivate their definition of the addition of vectors by considering the addition of displacement vectors. In a math class, we simply take the definition as given (even though the definition is motivated by physical experience). In Figure 3-2, a particle moves from point A to point B with corresponding displacement vector $AB = \vec{a}$, and then moves from point B to point C with corresponding displacement vector $BC = \vec{b}$. The net displacement vector when the particle moves from starting point A to final point C is $AC = \vec{s}$. In this way, the vector $AC = \vec{s}$ is defined as the *vector sum* or *resultant* of the vectors

$AB = \vec{a}$ and $BC = \vec{b}$, denoted $\vec{s} = \vec{a} + \vec{b}$.

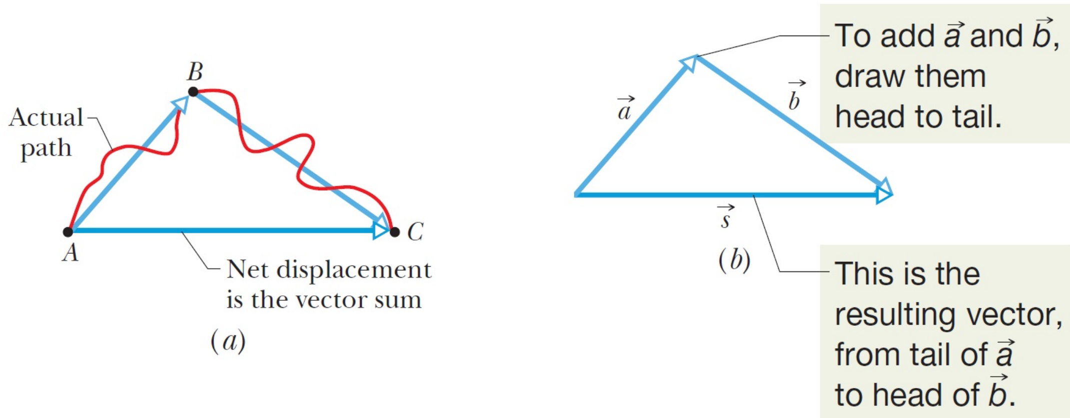

Figure 3-2 (*a*) *AC* is the vector sum of the vectors *AB* and *BC*. (*b*) The same vectors relabeled.

So to add two vectors $\vec{a}$ and $\vec{b}$ (geometrically), we draw $\vec{a}$ as an arrow with the appropriate magnitude and direction (as some location), then place the tail of $\vec{b}$ at the head of $\vec{a}$ and draw an arrow of the appropriate magnitude and direction to represent $\vec{b}$. The sum $\vec{s} = \vec{a} + \vec{b}$ is then the vector represented by an arrow from the tail of $\vec{a}$ to the tail of $\vec{b}$. This arrow determines the magnitude and direction of $\vec{s}$.

Note/"Theorem." We now make some claims about the algebraic interaction of vectors. Given our geometric definition of vector addition, we can justify these claims with pictures. Once we express vectors in terms of components below, we can give much mathematically cleaner proofs; see my online notes for Linear Algebra (MATH 2010) on Section 1.1. Vectors in Euclidean Spaces and notice Theorem 1.1, Properties of Vector Algebra in $\mathbb{R}^n$. We claim:

(3-2) The Commutative Law: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

(3-3) The Associative Law: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

We justify these claims with Figures 3-3 and 3-4.

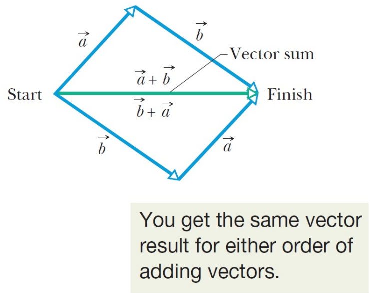

Figure 3-3 The two vectors $\vec{a}$ and $\vec{b}$ can be added in either order; see Eq. 3-2.

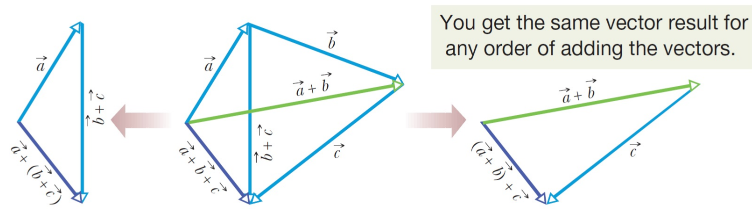

Figure 3-4 The three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be grouped in any way as they are added; see Eq. 3-3.

Note/Definition. For now, we define vector $-\vec{b}$ as the vector with the same magnitude as $\vec{b}$ but with opposite direction (see Figure 3-5). Notice that $\vec{b}+(-\vec{b}) = \vec{0}$ (a sum of two vectors is a vector, so we must have a vector on the right-hand side of this equation). Another unsurprising identity is:

(3-4) Vector Subtraction: $\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

We justify this with Figure 3-6.

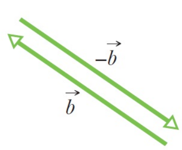

Figure 3-5 The vectors $\vec{b}$ and $-\vec{b}$ have the same magnitude and opposite directions.

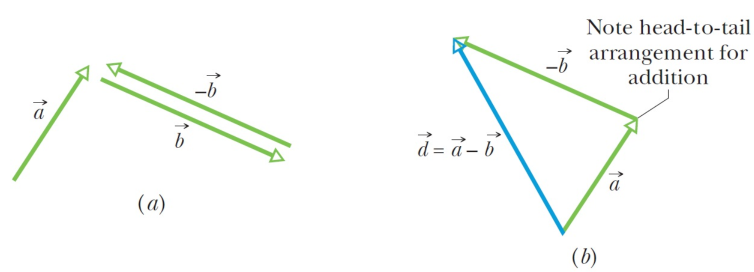

Figure 3-6 (*a*) Vectors $\vec{a}$, $\vec{b}$, and $-\vec{b}$. (*b*) To subtract vector $\vec{b}$ add vector $-\vec{b}$ to vector $\vec{a}$.

Note/Definition. For vectors in the plane, we introduce a horizontal x-axis and a vertical y-axis, as usual. A *component* of a vector is the projection of the vector onto an axis. We denote the component that results from projection of vector $\vec{a}$ onto the x-axis as a_x (called the x *component*), and the component that results from projection onto the y-axis as a_y (called the y *component*). This is illustrated in Figure 3-7. The process of finding the components of a vector is called *resolving the vector*.

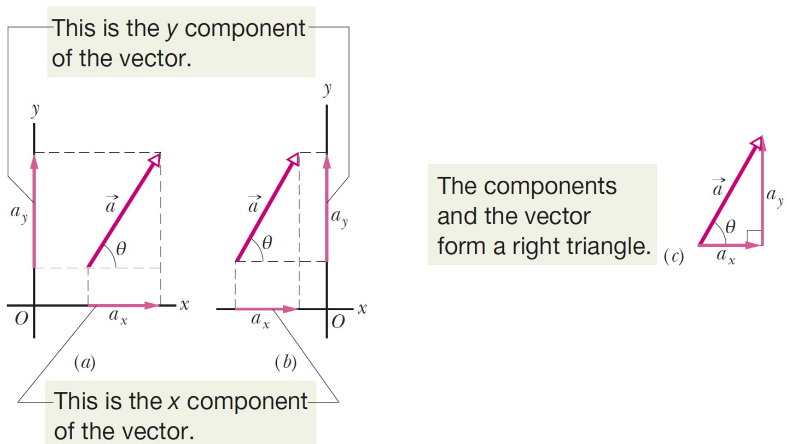

Figure 3-7 (a) The components a_x and a_y of vector $\vec{a}$. (b) The components are unchanged if the vector is shifted, as long as the magnitude and orientation are maintained. (c) The components form the legs of a right triangle whose hypotenuse is the magnitude of the vector.

Notice that Halliday and Resnick seem to label treat the components a_x and a_y as if they are vectors if Figure 3-7. This is an inconsistent use of the term "component"!!! They repeat this in Figure 3-8 and in their statement that "...we can reconstruct a vector from its components: we arrange those components *head to tail*. The we complete a right triangle with the vector forming the hypotenuse from the tail of one component to the head of the other component." (See page 43.) This is misleading and, strictly speaking, incorrect. Since a_x and a_y are scalars (that's the reason we don't draw little arrows over them), they do not have heads or tails! This is resolved using unit vectors in the next section.

Note. With θ as the angle vector $\vec{a}$ makes with the positive x-axis (as in Figure 3-7; θ could be taken to be between 0 and 2π, say) then the components of $\vec{a}$ are given by:

$$a_x = a\cos\theta \text{ and } a_y = a\sin\theta, \qquad (3\text{-}5)$$

where a is the magnitude of vector $\vec{a}$. Notice that the components of $\vec{a}$ may be negative. We can think of angle θ as giving the direction of $\vec{a}$ so that the magnitude and direction idea is explicitly given by a and θ. Relationships relating the components a_x, a_y, the magnitude a, and the direction θ include:

$$a = \sqrt{a_x^2 + a_y^2} \text{ and } \tan\theta = \frac{a_y}{a_x}, \qquad (3\text{-}6)$$

where $a_x \neq 0$. Notice that we cannot conclude that $\theta = \tan^{-1}(a_y/a_x)$, but we can use the tangent inverse function to determine θ, provided we take into consideration the signs of a_x and a_y (because the period of the tangent function is only π; see Figure 3-12).

Question 3.2. The two vectors shown in Figure 3-21 lie in an xy-plane. What are the signs of the x and y components, respectively, of **(a)** $\vec{d_1} + \vec{d_2}$, **(b)** $\vec{d_1} - \vec{d_2}$, and **(c)** $\vec{d_2} - \vec{d_1}$?

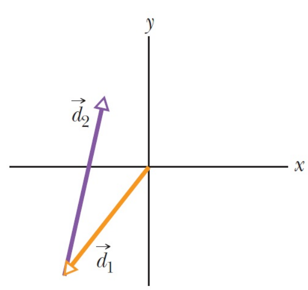

Figure 3-21 Question 2.

Section 3.2. Unit Vectors, Adding Vectors by Components

Definition. A *unit vector* is a vector that has magnitude one. We denote the unit vectors in the positive directions of the x, y, and z axes as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ respectively. These are called the *standard unit vectors.* Halliday and Resnick uses a slightly different notation and denotes them as $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$.

Note. The standard basis vectors in an xyz-coordinate system are given in Figure 3-13 (where the vectors have been placed in *standard position* with their tails at the origin). The arrangement of the axes in Figure 3-13 form a *right-handed coordinate system*, so called because if you curl the fingers of your right hand from the positive x-axis to the positive x-axis then your thumb points along the positive z-axis.

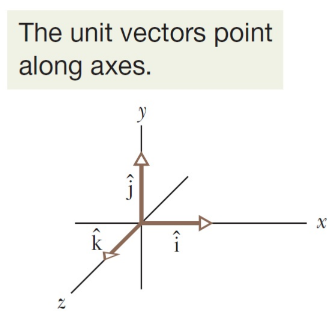

Figure 3-13 Unit vectors $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$ define the directions of a right-handed coordinate system.

Note. We can relate the components of a vector in the plane to the vector itself as follows:

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} \text{ and } \vec{b} = b_x\hat{\imath} + b_y\hat{\jmath}. \qquad (3\text{-}7,\ 3\text{-}8)$$

Figure 3-14 illustrates the relationship between $\vec{a}$, a_x, and a_y, and the relationship between $\vec{b}$, b_x, b_y, and θ.

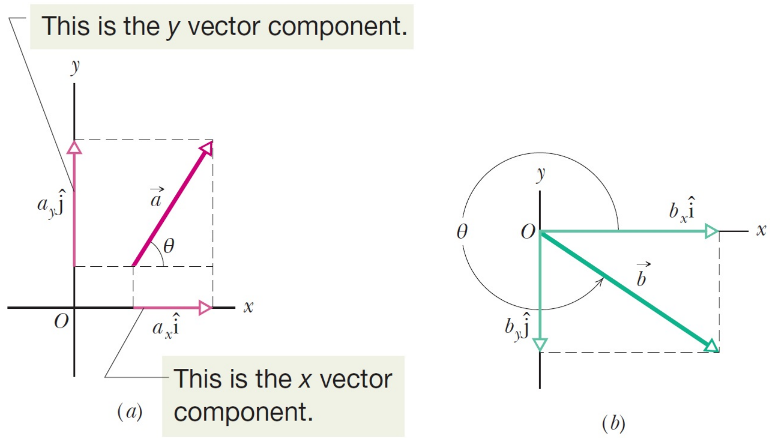

Figure 3-14 (*a*) The vector components of vector $\vec{a}$. (*b*) The vector components of vector $\vec{b}$.

Note. Notice that our geometric definition of vector addition (Definition 3.A) implies that we can add two vectors along the x-axis (which must be of the forms $a\hat{\imath}$ and $b\hat{\imath}$) by adding the components a and b to get $a\hat{\imath}+b\hat{\imath} = (a+b)\hat{\imath}$. Similar results hold for adding two vectors along the y-axis, or two vectors along the z-axis. So if we want to find the sum of two vectors $\vec{a}$ and $\vec{b}$ in three dimensions, we can write them in terms of their components and then add the components. We then have

$$\vec{r} = \vec{a}+\vec{b} = (a_x\hat{\imath}+a_y\hat{\jmath}+a_z\hat{k})+(b_x\hat{\imath}+b_y\hat{\jmath}+b_z\hat{k}) = (a_x+b_x)\hat{\imath}+(a_y+b_y)\hat{\jmath}+(a_z+b_z)\hat{k},$$

where we have used commutivity and associativity. We now see that the components of a sum of two vectors are the sum of the components of the two vectors. That is, for $\vec{r} = \vec{a}+\vec{b}$ we have $r_x = a_x+b_x$, $r_y = a_y+b_y$, and $r_z = a_z+b_z$. This is how we will commonly add vectors.

Question 3.6. Find two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^2$ (and their components) such that **(a)** $\vec{a}+\vec{b} = \vec{c}$ and $a+b=c$; **(b)** $\vec{a}+\vec{b} = \vec{a}-\vec{b}$; **(c)** $\vec{a}+\vec{b} = \vec{c}$ and $a^2+b^2 = c^2$.

Section 3.3. Multiplying Vectors

Note. We now describe three types of multiplication involving vectors. Each is very different from the other. Be careful when dealing with vector multiplication and make sure that you are using the correct type of product and getting the correct type of result (sometimes these products will be scalars and sometimes they will be vectors).

Definition. Let $\vec{a} \in \mathbb{R}^2$, $\vec{b} \in \mathbb{R}^3$, and let $s \in \mathbb{R}$ be a scalar. Geometrically, we define *scalar multiplication*, $s\vec{a}$, as the vector with magnitude $|s|$ times the magnitude of $\vec{a}$ with the same direction as $\vec{a}$ if $s > 0$ and the opposite direction as $\vec{a}$ if $s < 0$. We define scalar multiplication $s\vec{b}$ similarly. Algebraically, for $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$ and $\vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$, define $s\vec{a} = (sa_x)\hat{\imath} + (sa_y)\hat{\jmath}$ and $s\vec{b} = (sb_x)\hat{\imath} + (sb_y)\hat{\jmath} + (sb_z)\hat{k}$. (Notice that these two definitions are consistent with each other.)

Definition. Let $\vec{a}, \vec{b} \in \mathbb{R}^2$ or $\vec{a}, \vec{b} \in \mathbb{R}^3$. The *scalar product* (or *dot product* or *inner product*), denoted $\vec{a} \cdot \vec{b}$, is $\vec{a} \cdot \vec{b} = ab\cos\varphi$, where a is the magnitude of $\vec{a}$, b is the magnitude of vector $\vec{b}$, and φ is the angle between $\vec{a}$ and $\vec{b}$.

Note. Notice that a scalar product of two *vectors* is a *scalar* (thus the name!). Finding an angle between two vertices is potentially a geometric challenge. It turns out, that we are about to see an algebraic way to compute dot products. We can then use dot products to find angles, instead of the other way around. Geometrically, you should often think of dot products as some type of projection.

Notice that for any vector $\vec{a}$, then angle between vector $\vec{a}$ and itself is $0^\circ = 0$ so that $\vec{a} \cdot \vec{a} = aa \cos 0 = a^2(1) = a^2$. From the definition, we have that $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$; that is, the scalar product is commutative. You might be concerned with the whether the angle "between" $\vec{a}$ and $\vec{b}$ is the same as the angle "between" $\vec{b}$ and $\vec{a}$; it is (that's why the term "between" is used), and if we did put a sign on the angle then it would be absorbed by the cosine function.

Note. The scalar product distributes over vector addition. Also, scalars can be "pulled through" a scalar product: $\vec{a} \cdot (s\vec{b}) = (s\vec{a}) \cdot \vec{b} = s(\vec{a} \cdot \vec{b})$. We accept these properties; for a rigorous development of the scalar product, see my online notes for Linear Algebra (MATH 2010) on Section 1.2. The Norm and Dot Product. These properties allow us to compute scalar products in terms of components:

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= (a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}) \cdot (b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}) \\
&= (a_x\hat{\imath}) \cdot (b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}) + (a_y\hat{\jmath}) \cdot (b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}) + (a_z\hat{k}) \cdot (b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}) \\
&\quad \text{by distribution of the scalar product over vector addition} \\
&= (a_x\hat{\imath}) \cdot (b_x\hat{\imath}) + (a_x\hat{\imath}) \cdot (b_y\hat{\jmath}) + (a_x\hat{\imath}) \cdot (b_z\hat{k}) + (a_y\hat{\jmath}) \cdot (b_x\hat{\imath}) + (a_y\hat{\jmath}) \cdot (b_y\hat{\jmath}) \\
&\quad + (a_y\hat{\jmath}) \cdot (b_z\hat{k}) + (a_z\hat{k}) \cdot (b_x\hat{\imath}) + (a_z\hat{k}) \cdot (b_y\hat{\jmath}) + (a_z\hat{k}) \cdot (b_z\hat{k}) \\
&\quad \text{by distribution of the scalar product over vector addition} \\
&= (a_xb_x)(\hat{\imath} \cdot \hat{\imath}) + (a_xb_y)(\hat{\imath} \cdot \hat{\jmath}) + (a_xb_z)(\hat{\imath} \cdot \hat{k}) + (a_yb_x)(\hat{\jmath} \cdot \hat{\imath}) + (a_yb_y)(\hat{\jmath} \cdot \hat{\jmath}) \\
&\quad + (a_yb_z)(\hat{\jmath} \cdot \hat{k}) + (a_zb_x)(\hat{k} \cdot \hat{\imath}) + (a_zb_y)(\hat{k} \cdot \hat{\jmath}) + (a_zb_z)(\hat{k} \cdot \hat{k}) \\
&\quad \text{since scalars pull through} \\
&= (a_xb_x)(1) + (a_xb_y)(0) + (a_xb_z)(0) + (a_yb_x)(0) + (a_yb_y)(1) \\
&\quad + (a_yb_z)(0) + (a_zb_x)(0) + (a_zb_y)(0) + (a_zb_z)(1)
\end{aligned}$$

$$\begin{aligned}
&\quad \text{since } \hat\imath\cdot\hat\imath=\hat\jmath\cdot\hat\jmath=\hat k\cdot\hat k=1 \text{ and the other dot products}\\
&\quad \text{are 0 since } \hat\imath,\hat\jmath,\hat k \text{ are pairwise perpendicular (so } \varphi=0)\\
&= a_x b_x + a_y b_y + a_z b_z.
\end{aligned}$$

Therefore, scalar products are easy to calculate in terms of components:

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z. \tag{3-23}$$

Note. A scalar product is a product of two quantities: (1) the magnitude of one of the vectors, and (2) the scalar component of the second vector along the direction of the first vector. In Figure 3-18(b), $\vec{a}$ has a scalar component $a\cos\varphi$ along the direction of $\vec{b}$ and $\vec{b}$ has a scalar component $b\cos\varphi$ along the direction of $\vec{a}$.

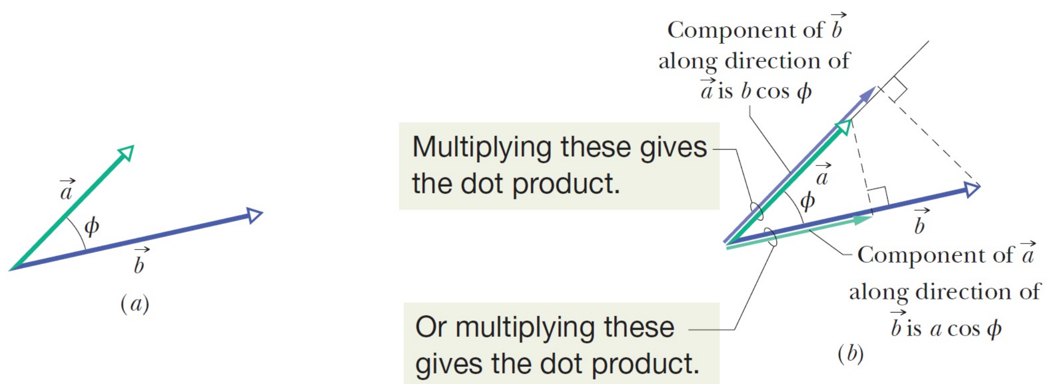

Figure 3-18 (a) Two vectors $\vec{a}$ and $\vec{b}$, with an angle ϕ between them. (*b*) Each vector has a component along the direction of the other vector.

Definition. Let $\vec{a},\vec{b}\in\mathbb{R}^3$. If $\vec{a}$ and $\vec{b}$ are parallel (i.e., they have the same direction), antiparallel (i.e., they have the opposite direction), or either is $\vec{0}$, then the *vector product* of $\vec{a}$ and $\vec{b}$, denoted $\vec{a}\times\vec{b}$, is the zero vector $\vec{0}$. For other vectors

$\vec{a}, \vec{b} \in \mathbb{R}^3$ the *vector product* (or *cross product*) of $\vec{a}$ and $\vec{b}$, denoted $\vec{a} \times \vec{b}$, is the vector $\vec{c}$ whose magnitude is $c = ab \sin \varphi$ where φ is the smaller of the two angles between $\vec{a}$ and $\vec{b}$ (i.e., we choose φ to be between 0° and 180°; notice we then have $\sin \varphi \geq 0$), and whose direction is determined by the right-hand rule of Figure 3-19(a). That is, $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ and in the direction of your thumb when you curl the fingers of your right hand from $\vec{a}$ to $\vec{b}$ where $\vec{a}$ and $\vec{b}$ are placed with their tails at the same point.

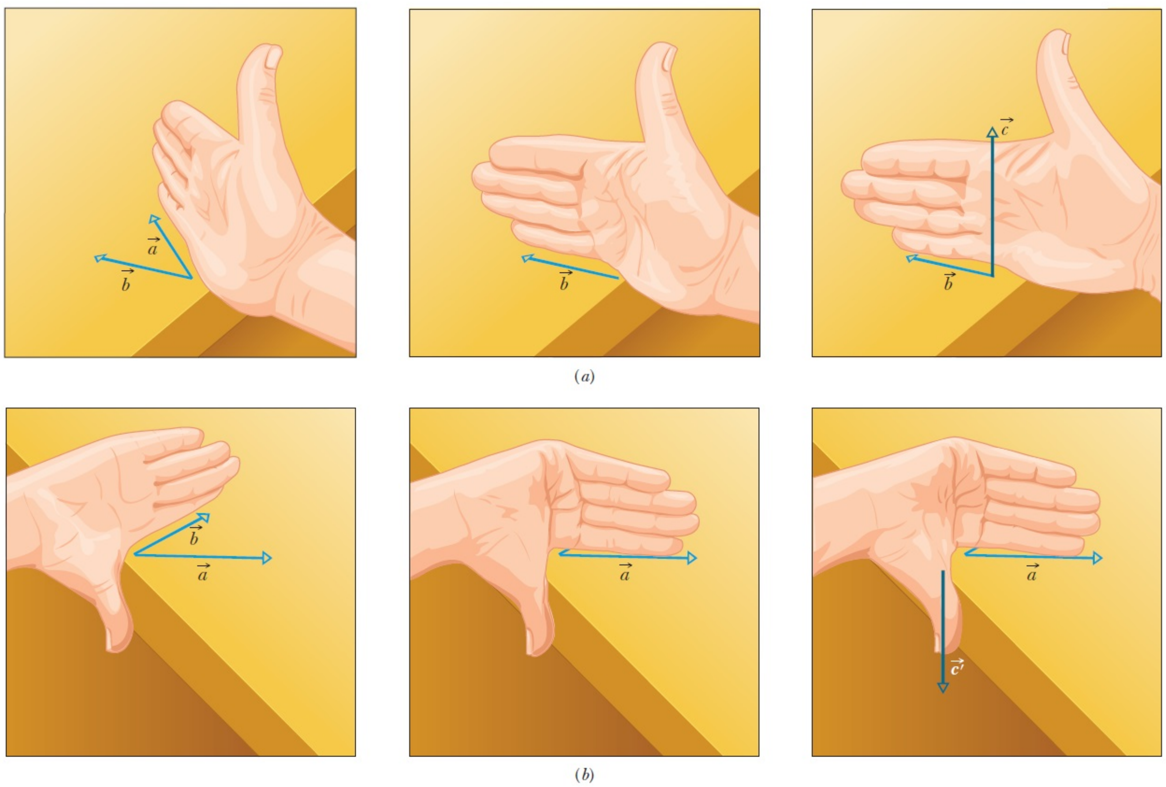

Figure 3-19 Illustration of the right-hand rule for vector products. (*a*) Sweep vector $\vec{a}$ into vector $\vec{b}$ with the fingers of your right hand. Your outstretched thumb shows the direction of vector $\vec{c} = \vec{a} \times \vec{b}$. (*b*) Showing that $\vec{b} \times \vec{a}$ is the reverse of $\vec{a} \times \vec{b}$.

Note. From the right-hand rule, we see that $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$; see also Figure 3-19(b). That is, the vector product is not commutative (this property is sometimes called *anticommutative*). By the definition we see that $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} =$

$\vec{0}$ (notice that a vector product is a *vector*, so the vector notation is necessary; Halliday and Resnick misuse the notation in these cases). We also have by the definition that $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, and $\hat{k} \times \hat{\imath} = \hat{\jmath}$. The vector product distributes over vector addition. Also, scalars can be "pulled through" a vector product: $\vec{a} \times (s\vec{b}) = (s\vec{a}) \times \vec{b} = s(\vec{a} \times \vec{b})$. We accept these properties; for more details on the vector product, see my online notes for Linear Algebra (MATH 2010) on Section 4.1. Areas, Volumes, and Cross Products or my online notes for Calculus 3 (MATH 2110) on Section 12.4. The Cross Product. This allows us to compute the cross product in terms of components:

$$\begin{aligned}
\vec{a} \times \vec{b} &= (a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}) \times (b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}) \\
&= (a_xb_x)(\hat{\imath} \times \hat{\imath}) + (a_xb_y)(\hat{\imath} \times \hat{\jmath}) + (a_xb_z)(\hat{\imath} \times \hat{k}) + (a_yb_x)(\hat{\jmath} \times \hat{\imath}) + (a_yb_y)(\hat{\jmath} \times \hat{\jmath}) \\
&\quad +(a_yb_z)(\hat{\jmath} \times \hat{k}) + (a_zb_x)(\hat{k} \times \hat{\imath}) + (a_zb_y)(\hat{k} \times \hat{\jmath}) + (a_zb_z)(\hat{k} \times \hat{k}) \\
&\quad\ \text{by distribution and scalars pull through} \\
&= (a_xb_x)(\hat{\imath} \times \hat{\imath}) + (a_xb_y)(\hat{\imath} \times \hat{\jmath}) + (a_xb_z)(-\hat{k} \times \hat{\imath}) + (a_yb_x)(-\hat{\imath} \times \hat{\jmath}) + (a_yb_y)(\hat{\jmath} \times \hat{\jmath}) \\
&\quad +(a_yb_z)(\hat{\jmath} \times \hat{k}) + (a_zb_x)(\hat{k} \times \hat{\imath}) + (a_zb_y)(-\hat{\jmath} \times \hat{k}) + (a_zb_z)(\hat{k} \times \hat{k}) \\
&\quad\ \text{by anticommutivity} \\
&= (a_xb_x)(\vec{0}) + (a_xb_y)(\hat{k}) + (a_xb_z)(-\hat{\jmath}) + (a_yb_x)(-\hat{k}) + (a_yb_y)(\vec{0}) \\
&\quad +(a_yb_z)(\hat{\imath}) + (a_zb_x)(\hat{\jmath}) + (a_zb_y)(-\hat{\imath}) + (a_zb_z)(\vec{0}) \\
&\quad\ \text{by the vector products of } \hat{\imath},\ \hat{\jmath}, \text{ and } \hat{k} \\
&= (a_yb_z - b_ya_z)\hat{\imath} + (a_zb_x - b_za_x)\hat{\jmath} + (a_xb_y - b_xa_y)\hat{k}.
\end{aligned}$$

Therefore, vector products can be calculated in terms of components:

$$\vec{a}\times\vec{b} = (a_yb_z - b_ya_z)\hat{\imath} + (a_zb_x - b_za_x)\hat{\jmath} + (a_xb_y - b_xa_y)\hat{k}. \tag{3-27}$$

Problem 3-1.12. A car is driven east for a distance of 50 km, then north for 30 km, and then in a direction 30° east of north for 25 km. Sketch the vector diagram and determine **(a)** the magnitude and **(b)** the angle of the car's total displacement from its starting point.

Problem 3-2.20. An explorer is caught in a whiteout (in which the snowfall is so thick that the ground cannot be distinguished from the sky) while returning to base camp. He was supposed to travel due north for 5.6 km, but when the snow clears, he discovers that he actually traveled 7.8 km at 50° north of due east. **(a)** How far and **(b)** in what direction must he now travel to reach base camp?

Problem 3-3.40. Displacement $\vec{d_1}$ is in the yz plane 63.0° from the positive direction of the y axis, has a positive z component, and has a magnitude of 4.50 m. Displacement $\vec{d_2}$ is in the xz plane 30.0° from the positive direction of the x axis, has a positive z component, and has magnitude 1.40 m. What are **(a)** $\vec{d_1} \cdot \vec{d_2}$, **(b)** $\vec{d_1} \times \vec{d_2}$, and **(c)** the angle between $\vec{d_1}$ and $\vec{d_2}$?

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