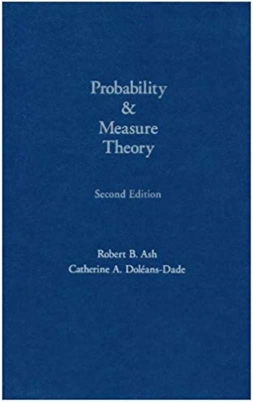
Real Analysis

Chapter 1. Fundamentals of Measure and Integration Theory

1.4. Lebesgue-Stieltjes Measures and Distribution Functions—Proofs of Theorems



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Proof. Since a measure is nonnegative then for $a < b$ we have $F(b) - F(a) = \mu((a, b]) \geq 0$, so that $F(a) \leq F(b)$ and F is increasing, as claimed.

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Let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary sequence where $x_1 > x_2 > x_3 > \cdots$ and $\lim_{n\to\infty} x_n = x_0$. Then the sequence of sets $(x_0, x_1] \supset (x_0, x_2] \supset (x_0, x_3] \supset \cdots$ is descending, so by the continuity of measure (Proposition 17.2(ii) of Royden and Fitzpatrick) we have

$$\lim_{n\to\infty}(F(x_n) - F(x_0)) = \lim_{n\to\infty} \mu((x_0, x_n]) = \mu\left(\lim_{n\to\infty}(x_0, x_n]\right) = \mu(\varnothing) = 0.$$

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Theorem 1.4.2 (continued)

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Proof (continued). Since sequence $\{x_n\}_{n=1}^{\infty}$ is an arbitrary monotone decreasing sequence approaching x, then $\lim_{x \to x_0^+}(F(x) - F(x_0)) = 0$, or $\lim_{x \to x_0^+} F(x) = F(x_0)$; that is F is right-continuous at x_0. Since x_0 is an arbitrary real number, then $F : \mathbb{R} \to \mathbb{R}$ so right continuous, as claimed. □

Lemma 1.4.3

Lemma 1.4.3. For μ defined above on field $\mathcal{F}_0(\overline{\mathbb{R}})$, μ is countably additive. That is, for $A_1, A_2, \ldots$ disjoint sets in $\mathcal{F}_0(\overline{\mathbb{R}})$ with $\cup_{n=1}^{\infty} A_n \in \mathcal{F}_0(\overline{\mathbb{R}})$ we have $\mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

Proof.

Case 1. First, suppose $F(\infty) - F(-\infty) < \infty$ so that μ is finite. Let $A_1 \supset A_2 \supset A_3 \supset \cdots$ be a decreasing sequence of sets in $\mathcal{F}_0(\overline{\mathbb{R}})$ such that $\lim_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n = \varnothing$. Let $\varepsilon > 0$. For any $(a, b]$, since F is right continuous at $x = a$, there is a $\delta > 0$ such that if $|a - a'| < \delta$ and $b > a^prime > a$ then $F(a') - F(a) < \varepsilon$. Then $(a', b] \subset (a, b]$ and

$$\mu((a, b]) = \mu((a', b]) = F(b) - F(a) - (F(b) - f(a')) = F(a') - F(a) < \varepsilon.$$

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$$\mu((a, b]) = \mu((a', b]) = F(b) - F(a) - (F(b) - f(a')) = F(a') - F(a) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then for each $A_n \in \mathcal{F}_0(\overline{\mathbb{R}})$ above, there is $B_n \in \mathcal{F}_0(\overline{\mathbb{R}})$ with $B_n \subset A_n$ (and $\overline{B_n} \subset A_n$) and $\mu(A_n) - \mu(B_n) < \varepsilon/2^n$ (since A_n consists of a finite union of intervals of the form $(a, b]$ so we can find appropriate corresponding $(a', b] \subset (a, b]$ such that $[a', b] \subset (a, b]$; this is where we also use the fact that μ is a finite measure).

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Case 1. First, suppose $F(\infty) - F(-\infty) < \infty$ so that μ is finite. Let $A_1 \supset A_2 \supset A_3 \supset \cdots$ be a decreasing sequence of sets in $\mathcal{F}_0(\overline{\mathbb{R}})$ such that $\lim_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n = \varnothing$. Let $\varepsilon > 0$. For any $(a, b]$, since F is right continuous at $x = a$, there is a $\delta > 0$ such that if $|a - a'| < \delta$ and $b > a^prime > a$ then $F(a') - F(a) < \varepsilon$. Then $(a', b] \subset (a, b]$ and

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Lemma 1.4.3 (continued 1)

Proof (continued). Since $\cap_{n=1}^{\infty} A_n = \varnothing$ then $\cap_{n=1}^{\infty} \overline{B}_n = \varnothing$. Since each $\overline{B}_n$ is closed, then each $\overline{B}_n^c = \overline{\mathbb{R}} \setminus \overline{B}_n$ is open (in the order topology) and so $\left(\cap_{n=1}^{\infty} \overline{B}_n\right)^c = \varnothing^c$ or $\cup_{n=1}^{\infty} \overline{B}_n^c = \overline{\mathbb{R}}$ (by De Morgan's Laws). Since $\overline{\mathbb{R}}$) is compact and $\{\overline{B}_n^c\}$ is an open cover of $\overline{\mathbb{R}}$, then there is some finite subcover of $\overline{\mathbb{R}}$, say $\overline{B}_1^c, \overline{B}_2^c, \ldots, \overline{B}_n^c$ (that is, $\{\overline{B}_k^c\}_{n=1}^{n}$ is a cover of $\overline{\mathbb{R}}$ for some $n \in \mathbb{N}$). Then $\cup_{k=1}^{n} \overline{B}_k^c = \overline{\mathbb{R}}$ and $\cap_{k=1}^{n} \overline{B}_k = \varnothing$. Now

$$\begin{aligned} \mu(A_n) &= \mu\left(\left(A_n \setminus \cap_{k=1}^{n} B_k\right) \,\dot\cup\, \left(\cap_{k=1}^{n} B_k\right)\right) \\ &= \mu\left(A_n \setminus \cap_{k=1}^{n} B_k\right) + \mu\left(\cap_{k=1}^{n} B_k\right) \\ &= \mu\left(A_n \setminus \cap_{k=1}^{n} B_k\right) \text{ since } \cap_{k=1}^{n} B_k \subset \cap_{k=1}^{n} \overline{B}_k = \varnothing. \end{aligned}$$

Now

$$\begin{aligned} A_n \setminus \cap_{k=1}^{n} B_k &= \cup_{k=1}^{n}(A_n \setminus B_k) = \cup_{k=1}^{n}(A_n \cap B_k^c) \\ &= \cup_{k=1}^{n}(A_k \cap B_k^c) \text{ since } A_n \subset A_k \text{ for } k = 1, 2, \ldots, n \\ &= \cup_{k=1}^{n}(A_k \setminus B_k), \end{aligned}$$

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$$\begin{aligned} \mu(A_n) &= \mu\left((A_n \setminus \cap_{k=1}^n B_k) \,\dot\cup\, (\cap_{k=1}^n B_k)\right) \\ &= \mu\left(A_n \setminus \cap_{k=1}^n B_k\right) + \mu\left(\cap_{k=1}^n B_k\right) \\ &= \mu\left(A_n \setminus \cap_{k=1}^n B_k\right) \text{ since } \cap_{k=1}^n B_k \subset \cap_{k=1}^n \overline{B}_k = \varnothing. \end{aligned}$$

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Lemma 1.4.3 (continued 2)

Proof (continued). ...so we have

$$\begin{aligned} \mu(A_n) &= \mu\left(\cup_{k=1}^n (A_k \setminus B_k)\right) \text{ since } F \text{ is increasing then } \mu \text{ is monotone} \\ &\le \sum_{k=1}^n \mu(A_k \setminus B_k) \text{ by Exercise 1.4.B(i)} \\ &< \sum_{k=1}^n \frac{\varepsilon}{2^k} < \varepsilon. \end{aligned}$$

So for given $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\mu(A_n) < \varepsilon$ (and by monotonicity, $\mu(A_k) \le \mu(A_n) < \varepsilon$ for $k \ge n$). Therefore, $\lim_{n\to\infty} \mu(A_n) = 0$.

Now let $C_1, C_2, \ldots$ be disjoint sets in $\mathcal{F}_0(\overline{\mathbb{R}})$ such that $C = \dot\cup_{n=1}^\infty C_n \in \mathcal{F}_0(\overline{\mathbb{R}})$. Let $A_n = C \setminus \cup_{k=1}^n C_n$ so that $A_1 \supset A_2 \supset \cdots$ and $\lim_{n\to\infty} A_n = \lim_{n\to\infty} (C \setminus \cup_{k=1}^n C_n) = \varnothing$.

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Lemma 1.4.3 (continued 3)

Proof (continued). Since we now know that $\lim_{n\to\infty} \mu(A_n) = 0$ and we have

$$\begin{aligned}\mu(C) &= \mu\left((C \setminus \cup_{k=1}^n C_n) \mathbin{\dot\cup} (\cup_{k=1}^n C_n)\right)\\ &= \mu\left(C \setminus \dot\cup_{k=1}^n C_n\right) + \mu\left(\dot\cup_{k=1}^n C_n\right) \text{ since } \mu \text{ is}\\ &\quad\ \text{finite additive additive by hypothesis}\\ &= \mu(A_n) + \sum_{k=1}^n \mu(C_n) \text{ since } \mu \text{ is finite additive}\end{aligned}$$

so

$$\lim_{n\to\infty} \mu(C) = \lim_{n\to\infty} \left(\mu(A_n) + \sum_{k=1}^n \mu(C_n)\right)$$

$$= \lim_{n\to\infty} \mu(A_n) + \lim_{n\to\infty} \left(\sum_{k=1}^n \mu(C_n)\right) = 0 + \sum_{k=1}^\infty \mu(C_n),$$

or $\mu(C) = \mu\left(\dot\cup_{k=1}^\infty C_n\right) = \sum_{k=1}^\infty \mu(C_n)$. That is, μ is countable additive in the case that μ is a finite measure.

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Proof (continued). Since we now know that $\lim_{n\to\infty}\mu(A_n)=0$ and we have

$$\begin{aligned}
\mu(C) &= \mu\left((C\setminus\cup_{k=1}^n C_n)\,\dot\cup\,(\cup_{k=1}^n C_n)\right)\\
&= \mu\left(C\setminus\dot\cup_{k=1}^n C_n\right)+\mu\left(\dot\cup_{k=1}^n C_n\right) \text{ since } \mu \text{ is}\\
&\quad \text{finite additive additive by hypothesis}\\
&= \mu(A_n)+\sum_{k=1}^n \mu(C_n) \text{ since } \mu \text{ is finite additive}
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so

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or $\mu(C)=\mu\left(\dot\cup_{k=1}^\infty C_n\right)=\sum_{k=1}^\infty\mu(C_n)$. That is, μ is countable additive in the case that μ is a finite measure.

Lemma 1.4.3 (continued 4)

Proof (continued).
Case 2. Second, suppose $F(\infty) - F(-\infty) = \infty$. Define

$$F_n(x) = \begin{cases} F(n) & \text{for } x > n \\ F(x) & \text{for } |x| \leq n \\ F(-n) & \text{for } x < -n. \end{cases}$$

Let μ_n be the set function on $\mathcal{F}_0(\overline{\mathbb{R}})$ defined with $\mu_n((a,b]) = F_n(b) - F_n(a)$, as above. Then $\mu_n \leq \mu$ on $\mathcal{F}_0(\overline{\mathbb{R}})$ and $\lim_{n\to\infty} \mu_n = \mu$. Let $A_1, A_2, \ldots$ be disjoint sets in $\mathcal{F}_0(\overline{\mathbb{R}})$ such that $A = \cup_{n=1}^{\infty} A_n \in \mathcal{F}_0(\overline{\mathbb{R}})$. Then $\mu(A) = \mu\left(\cup_{n=1}^{\infty} A_n\right) \geq \sum_{n=1}^{\infty} \mu(A_n)$ by Exercise 1.4.B(ii). So if $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ then $\mu(A) = \infty$ and countable additivity holds in this case. So we can without loss of generality assume $\sum_{n=1}^{\infty} \mu(A_n) < \infty$.

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Lemma 1.4.3 (continued 5)

Proof (continued). Then

$$\begin{aligned} \mu(A) &= \lim_{n\to\infty} \mu_n(A) \text{ since } \lim_{n\to\infty} \mu_n = \mu \text{ on } \mathcal{F}_0(\overline{\mathbb{R}}) \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \mu_n(A_k) \right) \text{ by Case 1, since} \\ &\qquad A_k \text{ are disjoint and } \mu_n \text{ is finite.} \end{aligned}$$

Since $\sum_{k=1}^{\infty} \mu(A_k) < \infty$ and $\mu(A) \geq \sum_{n=1}^{\infty} \mu(A_n)$ by Exercise 1.4.B(ii), then

$$\begin{aligned} 0 &\leq \mu(A) - \sum_{k=1}^{\infty} \mu(A_k) = \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \mu_n(A_k) \right) - \sum_{k=1}^{\infty} \mu(A_k) \\ &= \lim_{n\to\infty} \sum_{k=1}^{\infty} (\mu_n(A_k) - \mu(A_k)) \leq 0 \text{ since } \mu_n \leq \mu \text{ on } \mathcal{F}_0(\overline{\mathbb{R}}). \end{aligned}$$

Therefore $\mu(A) = \mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ and countable additivity holds in the case that μ is an infinite measure. □

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Since $\sum_{k=1}^{\infty} \mu(A_k) < \infty$ and $\mu(A) \ge \sum_{n=1}^{\infty} \mu(A_n)$ by Exercise 1.4.B(ii), then

$$\begin{aligned}0 &\le \mu(A) - \sum_{k=1}^{\infty} \mu(A_k) = \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \mu_n(A_k) \right) - \sum_{k=1}^{\infty} \mu(A_k) \\ &= \lim_{n\to\infty} \sum_{k=1}^{\infty} (\mu_n(A_k) - \mu(A_k)) \le 0 \text{ since } \mu_n \le \mu \text{ on } \mathcal{F}_0(\overline{\mathbb{R}}).\end{aligned}$$

Therefore $\mu(A) = \mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ and countable additivity holds in the case that μ is an infinite measure. □

Lemma 1.4.7

Lemma 1.4.7. Let $a, b \in \mathbb{R}^3$. If $a \leq b$ (that is, the coordinates of a and b satisfy $a_i \leq b_i$ for $i = 1, 2, 3$), then

(a) $\mu((a,b]) = \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3 \mid a_1 < \omega_1 \leq b_1, a_2 < \omega_2 \leq b_2, a_3 < \omega_3 \leq b_3\}) = \Delta_{b_1 a_1}\Delta_{b_2 a_2}\Delta_{b_3 a_3} F(x_1, x_2, x_3)$ where

(b) $\Delta_{b_1 a_1}\Delta_{b_2 a_2}\Delta_{b_3 a_3} F(x_1, x_2, x_3) = F(b_1, b_2, b_3) - F(a_1, b_2, b_3) - F(b_1, a_2, b_3) - F(b_1, b_2, a_3) + F(a_1, a_2, b_3) + F(a_1, b_2, a_3) + F(b_1, a_2, a_3) - F(a_1, a_2, a_3)$.

Proof. We have

$\Delta_{b_3 a_3} F(a_1, x_2, x_3) = F(x_1, x_2, b_3) - F(x_1, x_2, a_3)$ by the definition of $\Delta_{b_3 a_3}$

$$\begin{aligned} &= \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \leq x_1, \omega_2 \leq x_2, \omega_3 \leq b_3\}) \\ &\quad -\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \leq x_1, \omega_2 \leq x_2, \omega_3 \leq a_3\}) \text{ by } (*) \\ &= \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \leq x_1, \omega_2 \leq x_2, a_3 < \omega_3 \leq b_3\}) \end{aligned}$$

by the additivity of measure μ.

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Lemma 1.4.7. Let $a, b \in \mathbb{R}^3$. If $a \leq b$ (that is, the coordinates of a and b satisfy $a_i \leq b_i$ for $i = 1, 2, 3$), then

(a) $\mu((a,b]) = \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3 \mid a_1 < \omega_1 \leq b_1, a_2 < \omega_2 \leq b_2, a_3 < \omega_3 \leq b_3\}) = \Delta_{b_1 a_1} \Delta_{b_2 a_2} \Delta_{b_3 a_3} F(x_1, x_2, x_3)$ where

(b) $\Delta_{b_1 a_1} \Delta_{b_2 a_2} \Delta_{b_3 a_3} F(x_1, x_2, x_3) = F(b_1, b_2, b_3) - F(a_1, b_2, b_3) - F(b_1, a_2, b_3) - F(b_1, b_2, a_3) + F(a_1, a_2, b_3) + F(a_1, b_2, a_3) + F(b_1, a_2, a_3) - F(a_1, a_2, a_3)$.

Proof. We have

$\Delta_{b_3 a_3} F(a_1, x_2, x_3) = F(x_1, x_2, b_3) - F(x_1, x_2, a_3)$ by the definition of $\Delta_{b_3 a_3}$

$$\begin{aligned} &= \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \leq x_1, \omega_2 \leq x_2, \omega_3 \leq b_3\}) \\ &\quad -\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \leq x_1, \omega_2 \leq x_2, \omega_3 \leq a_3\}) \text{ by } (*) \\ &= \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \leq x_1, \omega_2 \leq x_2, a_3 < \omega_3 \leq b_3\}) \end{aligned}$$

by the additivity of measure μ.

Lemma 1.4.7 (continued 1)

Proof (continued). Next,

$$\Delta_{b_2a_2}\Delta_{b_3a_3}F(a_1,x_2,x_3) = \Delta_{b_2a_2}(F(x_1,x_2,b_3) - F(x_1,x_2,a_3)) \text{ from above}$$

$$\begin{aligned}
&= \Delta_{b_2a_2}F(a_1,x_2,b_3) - \Delta_{b_2a_2}F(x_1,x_2,a_3) \text{ since } \Delta_{b_ia_i} \text{ is linear} \\
&= F(x_1,b_2,b_3) - F(x_1,a_2,b_3) - F(x_1,b_2,a_3) + F(x_1,a_2,a_3) \\
&\quad \text{ by the definition of } \Delta_{b_2a_2} \\
&= \mu(\{\omega = (\omega_1,\omega_2,\omega_3) \mid \omega_1 \le x_1, \omega_2 \le b_2, \omega_3 \le b_3\}) \\
&\quad -\mu(\{\omega = (\omega_1,\omega_2,\omega_3) \mid \omega_1 \le x_1, \omega_2 \le a_2, \omega_3 \le a_3\}) \\
&\quad -\mu(\{\omega = (\omega_1,\omega_2,\omega_3) \mid \omega_1 \le x_1, \omega_2 \le b_2, \omega_3 \le a_3\}) \\
&\quad +\mu(\{\omega = (\omega_1,\omega_2,\omega_3) \mid \omega_1 \le x_1, a_2 < \omega_2, \omega_3 \le a_3\}) \text{ by } (*) \\
&= \mu(\{\omega = (\omega_1,\omega_2,\omega_3) \mid \omega_1 \le x_1, a_2 < \omega_2 \le b_2, a_3 < \omega_3\}) \\
&\quad -\mu(\{\omega = (\omega_1,\omega_2,\omega_3) \mid \omega_1 \le x_1, a_2 < \omega_2 \le b_2, \omega_3 \le a_3\}) \\
&\quad \text{ by the additivity of measure } \mu
\end{aligned}$$

Lemma 1.4.7 (continued 1)

Proof (continued). Next,

$$\Delta_{b_2 a_2}\Delta_{b_3 a_3}F(a_1,x_2,x_3) = \Delta_{b_2 a_2}(F(x_1,x_2,b_3) - F(x_1,x_2,a_3)) \text{ from above}$$

$$\begin{aligned}
&= \Delta_{b_2 a_2}F(a_1,x_2,b_3) - \Delta_{b_2 a_2}F(x_1,x_2,a_3) \text{ since } \Delta_{b_i a_i} \text{ is linear} \\
&= F(x_1,b_2,b_3) - F(x_1,a_2,b_3) - F(x_1,b_2,a_3) + F(x_1,a_2,a_3) \\
&\quad \text{by the definition of } \Delta_{b_2 a_2} \\
&= \mu(\{\omega = (\omega_1,\omega_2,\omega_3) \mid \omega_1 \le x_1, \omega_2 \le b_2, \omega_3 \le b_3\}) \\
&\quad -\mu(\{\omega = (\omega_1,\omega_2,\omega_3) \mid \omega_1 \le x_1, \omega_2 \le a_2, \omega_3 \le a_3\}) \\
&\quad -\mu(\{\omega = (\omega_1,\omega_2,\omega_3) \mid \omega_1 \le x_1, \omega_2 \le b_2, \omega_3 \le a_3\}) \\
&\quad +\mu(\{\omega = (\omega_1,\omega_2,\omega_3) \mid \omega_1 \le x_1, a_2 < \omega_2, \omega_3 \le a_3\}) \text{ by } (*) \\
&= \mu(\{\omega = (\omega_1,\omega_2,\omega_3) \mid \omega_1 \le x_1, a_2 < \omega_2 \le b_2, a_3 < \omega_3\}) \\
&\quad -\mu(\{\omega = (\omega_1,\omega_2,\omega_3) \mid \omega_1 \le x_1, a_2 < \omega_2 \le b_2, \omega_3 \le a_3\}) \\
&\quad \text{by the additivity of measure } \mu
\end{aligned}$$

Lemma 1.4.7 (continued 2)

Proof (continued).

$$= \quad \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \leq x_1, a_2 < \omega_2 \leq b_2, a_3 < \omega_3 \leq b_3\})$$

by the additivity of measure μ

and

$$\Delta_{b_1 a_1} \Delta_{b_2 a_2} \Delta_{b_3 a_3} F(x_1, x_2, x_3) = \Delta_{b_1 a_1}(F(x_1, b_2, b_3) - F(x_1, a_2, b_3)$$
$$-F(x_1, b_2, a_3) + F(x_1, a_2, a_3)) \text{ from above}$$

$$= \quad \Delta_{b_1 a_1} F(x_1, b_2, b_3) - \Delta_{b_1 a_1} F(x_1, a_2, b_3) - \Delta_{b_1 a_1} F(x_1, b_2, a_3)$$
$$+\Delta_{b_1 a_1} F(x_1, a_2, a_3)) \text{ since } \Delta_{b_i a_i} \text{ is linear}$$

Lemma 1.4.7 (continued 2)

Proof (continued).

$$= \quad \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \leq x_1, a_2 < \omega_2 \leq b_2, a_3 < \omega_3 \leq b_3\})$$

by the additivity of measure μ

and

$$\Delta_{b_1 a_1}\Delta_{b_2 a_2}\Delta_{b_3 a_3} F(x_1, x_2, x_3) = \Delta_{b_1 a_1}(F(x_1, b_2, b_3) - F(x_1, a_2, b_3)$$

$$-F(x_1, b_2, a_3) + F(x_1, a_2, a_3)) \text{ from above}$$

$$= \quad \Delta_{b_1 a_1} F(x_1, b_2, b_3) - \Delta_{b_1 a_1} F(x_1, a_2, b_3) - \Delta_{b_1 a_1} F(x_1, b_2, a_3)$$

$$+\Delta_{b_1 a_1} F(x_1, a_2, a_3)) \text{ since } \Delta_{b_i a_i} \text{ is linear}$$

$$= \quad F(b_1, b_2, b_3) - F(a_1, b_2, b_3) - F(b_1, a_2, b_3) + F(a_1, a_2, b_3)$$

$$-F(b_1, b_2, a_3) + F(a_1, b_2, a_3) + F(b_1, a_2, a_3) - F(a_1, a_2, a_3)$$

by the definition of $\Delta_{b_1 a_1}$

Lemma 1.4.7 (continued 2)

Proof (continued).

$$= \quad \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \leq x_1, a_2 < \omega_2 \leq b_2, a_3 < \omega_3 \leq b_3\})$$

by the additivity of measure μ

and

$$\Delta_{b_1 a_1} \Delta_{b_2 a_2} \Delta_{b_3 a_3} F(x_1, x_2, x_3) = \Delta_{b_1 a_1}(F(x_1, b_2, b_3) - F(x_1, a_2, b_3) - F(x_1, b_2, a_3) + F(x_1, a_2, a_3)) \text{ from above}$$

$$\begin{aligned} &= \Delta_{b_1 a_1} F(x_1, b_2, b_3) - \Delta_{b_1 a_1} F(x_1, a_2, b_3) - \Delta_{b_1 a_1} F(x_1, b_2, a_3) \\ &\quad + \Delta_{b_1 a_1} F(x_1, a_2, a_3)) \text{ since } \Delta_{b_i a_i} \text{ is linear} \\ &= F(b_1, b_2, b_3) - F(a_1, b_2, b_3) - F(b_1, a_2, b_3) + F(a_1, a_2, b_3) \\ &\quad - F(b_1, b_2, a_3) + F(a_1, b_2, a_3) + F(b_1, a_2, a_3) - F(a_1, a_2, a_3) \\ &\quad \text{by the definition of } \Delta_{b_1 a_1} \end{aligned}$$

Lemma 1.4.7 (continued 3)

Proof (continued).

$$
\begin{aligned}
= \quad & \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \le b_1, \omega_2 \le b_2, \omega_3 \le b_3\}) \\
& -\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \le a_1, \omega_2 \le b_2, \omega_3 \le b_3\}) \\
& -\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \le b_1, \omega_2 \le a_2, \omega_3 \le b_3\}) \\
& +\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \le a_1, \omega_2 \le a_2, \omega_3 \le b_3\}) \\
& -\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \le b_1, \omega_2 \le b_2, \omega_3 \le a_3\}) \\
& +\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \le a_1, \omega_2 \le b_2, \omega_3 \le a_3\}) \\
& +\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \le b_1, \omega_2 \le a_2, \omega_3 \le a_3\}) \\
& -\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \le a_1, \omega_2 \le a_2, \omega_3 \le a_3\}) \text{ by } (*)
\end{aligned}
$$

Lemma 1.4.7 (continued 4)

Proof (continued).

$$\begin{aligned}
= \quad & \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \leq b_1, \omega_2 \leq b_2, \omega_3 \leq b_3\}) \\
& -\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \leq b_1, \omega_2 \leq a_2, \omega_3 \leq b_3\}) \\
& -\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \leq b_1, \omega_2 \leq b_2, a_3 < \omega_3\}) \\
& +\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \leq b_1, \omega_2 \leq a_2, \omega_3 \leq a_3\}) \\
& \text{ by the additivity of measure } \mu \text{ (combining pairs)} \\
= \quad & \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \leq b_1, a_2 < \omega_2 \leq b_2, \omega_3 \leq b_3\}) \\
& -\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \leq b_1, a_2 < \omega_2 \leq b_2, \omega_3 \leq a_3\}) \\
= \quad & \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \leq b_1, a_2 < \omega_2 \leq b_2, a_3 < \omega_3 \leq b_3\}) \\
& \text{ by the additivity of measure } \mu \text{ (combining pairs)}
\end{aligned}$$

□

Lemma 1.4.7 (continued 4)

Proof (continued).

$$\begin{aligned}
&= \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \le b_1, \omega_2 \le b_2, \omega_3 \le b_3\}) \\
&\quad -\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \le b_1, \omega_2 \le a_2, \omega_3 \le b_3\}) \\
&\quad -\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \le b_1, \omega_2 \le b_2, a_3 < \omega_3\}) \\
&\quad +\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \le b_1, \omega_2 \le a_2, \omega_3 \le a_3\}) \\
&\quad \text{ by the additivity of measure } \mu \text{ (combining pairs)} \\
&= \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \le b_1, a_2 < \omega_2 \le b_2, \omega_3 \le b_3\}) \\
&\quad -\mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \le b_1, a_2 < \omega_2 \le b_2, \omega_3 \le a_3\}) \\
&= \mu(\{\omega = (\omega_1, \omega_2, \omega_3) \mid a_1 < \omega_1 \le b_1, a_2 < \omega_2 \le b_2, a_3 < \omega_3 \le b_3\}) \\
&\quad \text{ by the additivity of measure } \mu \text{ (combining pairs)}
\end{aligned}$$

□