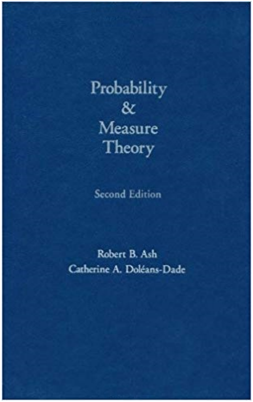
Real Analysis

Chapter 4. Basic Concepts of Probability

4.1. Independence—Proofs of Theorems



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Lemma 4.3.A

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Proof. Let $\{i_1, i_2, \ldots, i_k\}$ be distinct indices in I. Since the events A_i are independent, then

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k}).$$

Suppose we replace A_{i_j} with $A_{i_j}^c$. Let

$$B = A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k}.$$

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Then

$$\begin{aligned} P(B) &= P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k}) \\ &= P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_{j-1}})P(A_{i_{j+1}}) \cdots P(A_{i_k}) \end{aligned}$$

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Lemma 4.3.A (continued 1)

Proof (continued). Then

$$\begin{aligned} P(B \cap A_{i_j}^c) &= P(B \setminus A_{i_j}) = P(B \setminus (A_{i_j} \cap B)) \\ &= P(B) - P(A_{i_j} \cap B) \text{ by the excision principle} \\ &\quad \text{(Proposition 17.1(iii)) since } B \subset A_{i_j} \cap B \\ &\quad \text{and } P(B) \leq 1 < \infty \\ &= P(B) - P(A_{i_j})P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k}) \\ &\quad \text{since the } A_i \text{ are independent} \\ &= P(B) - P(A_{i_j})P(B) = P(B)(1 - P(A_{i_j})) = P(B)P(A_{i_j}^c). \end{aligned}$$

Therefore

$$P(B \cap A_{i_j}^c) = P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{j-1}} \cap A_{i_j}^c \cap A_{i_{j+1}} \cap \cdots \cap A_{i_k}) = P(B)P(A_{i_j})^c)$$

$$= P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_{j-1}})P(A_{i_j}^c)P(A_{i_{j+1}}) \cdots P(A_{i_k}),$$

so the claim holds. □

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$$= P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_{j-1}})P(A_{i_j}^c)P(A_{i_{j+1}}) \cdots P(A_{i_k}),$$

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