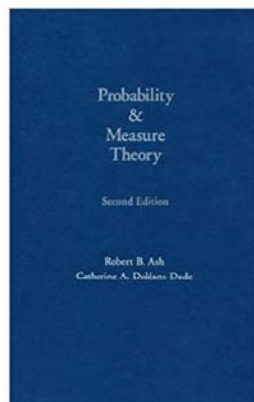


# Real Analysis

## Chapter 4. Basic Concepts of Probability

### 4.6. Random Variables—Proofs of Theorems



()

Real Analysis

February 2, 2019 1 / 10

Lemma 4.6.A

## Lemma 4.6.A

**Lemma 4.6.A.** Let  $X$  be a random variable on probability space  $(\Omega, \mathcal{F}, P)$ . Then the distribution function  $F$  of  $X$  is increasing and right-continuous. Also,

$$\lim_{x \rightarrow \infty} F(x) = 1 \text{ and } \lim_{x \rightarrow -\infty} F(x) = 0.$$

**Proof.** From (\*), for any  $a < b$  we have

$$F(b) - F(a) = P_X((a, b]) = P(\{\omega \mid X(\omega) \in (a, b]\}) \geq 0$$

since  $P$  is a measure. So  $F$  is increasing, as claimed. For  $b \in \mathbb{R}$ , we consider

$$\lim_{x \rightarrow b^+} F(x) = \lim_{h \rightarrow 0^+} F(b + h) = \lim_{h \rightarrow 0^+} P(\{\omega \mid X(\omega) \leq b + h\})$$

Since  $F$  is monotone increasing then one-sided limits exist, so we consider a sequence  $\{b + 1/n\} \rightarrow b^+$  to evaluate  $\lim_{x \rightarrow b^+} F(x)$ :

()

Real Analysis

February 2, 2019 3 / 10

Lemma 4.6.A

## Lemma 4.6.A, continued 1

**Proof (continued).**

$$\begin{aligned} \lim_{x \rightarrow b^+} F(x) &= \lim_{n \rightarrow \infty} P(\{\omega \mid X(\omega) \leq b + 1/n\}) \\ &= P\left(\lim_{n \rightarrow \infty} \{\omega \mid X(\omega) \leq b + 1/n\}\right) \text{ by the} \\ &\quad \text{Continuity of Measure (Proposition 17.2),} \\ &\quad \text{since } \{E_n\}_{n=1}^{\infty} \text{ where } E_n = \{\omega \mid X(\omega) \leq b + 1/n\} \\ &\quad \text{is a descending sequence of sets} \\ &= P(\cap_{n=1}^{\infty} \{\omega \mid X(\omega) \leq b + 1/n\}) \\ &= P(\{\omega \mid X(\omega) \leq b\}) = F(b). \end{aligned}$$

So  $F$  is right-continuous. Since  $P$  is a measure,  $P(\emptyset) = 0$  and since  $(\Omega, \mathcal{F}, P)$  is a probability space then  $P(\Omega) = 1$ .

()

Real Analysis

February 2, 2019 4 / 10

Lemma 4.6.A

## Lemma 4.6.A, continued 2

**Proof (continued).** Again by Continuity of Measure

$$\begin{aligned} 0 &= P(\emptyset) = P(\cap_{n=1}^{\infty} \{\omega \mid X(\omega) \leq -n\}) = P\left(\lim_{n \rightarrow \infty} \{\omega \mid X(\omega) \leq -n\}\right) \\ &= \lim_{n \rightarrow \infty} P(\{\omega \mid X(\omega) \leq -n\}) = \lim_{n \rightarrow \infty} F(-n) \\ &= \lim_{n \rightarrow -\infty} F(x) \text{ since } F \text{ is monotone increasing,} \end{aligned}$$

and

$$\begin{aligned} 1 &= P(\Omega) = P(\cup_{n=1}^{\infty} \{\omega \mid X(\omega) \geq n\}) \\ &= P\left(\lim_{n \rightarrow \infty} \{\omega \mid X(\omega) \geq n\}\right) \\ &= \lim_{n \rightarrow \infty} P(\{\omega \mid X(\omega) \geq n\}) = \lim_{n \rightarrow \infty} F(n) = \lim_{x \rightarrow \infty} F(x), \end{aligned}$$

as claimed. □

()

Real Analysis

February 2, 2019 5 / 10

## Lemma 4.6.B

**Lemma 4.6.B.** If  $F : \mathbb{R} \rightarrow [0, 1]$  is an increasing and right-continuous function with  $\lim_{x \rightarrow \infty} F(x) = 1$  and  $\lim_{x \rightarrow -\infty} F(x) = 0$ , then  $F$  is the distribution function of some random variable. Note: Though  $F$  is defined on  $\mathbb{R}$ , we denote  $\lim_{x \rightarrow \infty} F(x) = F(\infty)$  and  $\lim_{x \rightarrow -\infty} F(x) = F(-\infty)$ .

**Proof.** We take  $\Omega = \mathbb{R}$ ,  $\mathcal{F} = \mathcal{B}(\mathbb{R})$  (the Borel sets in  $\mathbb{R}$ ), and define  $X(\omega) = \omega$  for  $\omega \in \Omega$  (s  $X$  is the identity map on  $\Omega = \mathbb{R}$ ). We use  $F$  to define a probability measure  $P$ ; define  $P((a, b]) = F(b) - F(a)$  for all  $a, b \in \mathbb{R}$  and define  $P(\emptyset) = 0$ . Then  $P$  is defined on all  $(a, b] \subset \mathbb{R}$  and the set of all such  $(a, b]$  (along with  $\emptyset$ ) form a semiring  $\mathcal{S}$ . Notice that  $P$  is finitely additive on  $\mathcal{S}$ : if

$$(a, b] = (a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n]$$

(where, say,  $a_1 = a$ ,  $b_n = b$ , and  $a_i = b_{i-1}$  for  $i = 2, 3, \dots, n$ ) then...

## Lemma 4.6.B (continued 2)

**Proof (continued).** (Notice that the smallest  $\sigma$ -algebra containing all subsets of  $\mathbb{R}$  of the form  $(a, b]$  is  $\mathcal{B}(\mathbb{R})$ , by Exercise 1.36 in Royden and Fitzpatrick.) So  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P)$  is a probability space. For  $x \in \mathbb{R}$ ,

$$\begin{aligned} F(x) &= F(x) - F(-\infty) \text{ since } F(-\infty) = \lim_{y \rightarrow -\infty} F(y) = 0 \\ &= F(x) - \lim_{y \rightarrow -\infty} F(y) = \lim_{y \rightarrow -\infty} (F(x) - F(y)) \\ &= \lim_{y \rightarrow -\infty} P((y, x]) = \lim_{n \rightarrow -\infty} P((-n, x]) \text{ since } P \text{ is monotone} \\ &\quad \text{and } P((y, x]) \text{ is a decreasing function of } y \text{ so the limit can} \\ &\quad \text{be evaluated using any sequence of values } \{y_n\} \rightarrow -\infty \\ &= P\left(\lim_{n \rightarrow -\infty} (-n, x]\right) \text{ by Continuity of Measure since } \{(-n, x]\} \\ &\quad \text{is an increasing sequence of sets (for } n \text{ sufficiently large)} \\ &= P((-\infty, x]) = P(\omega \mid \omega \leq x) = P(\omega \mid X(\omega) \leq x) \text{ since } X(\omega) = \omega, \end{aligned}$$

so  $F$  is the distribution of random variable  $X$ , as claimed.  $\square$

## Lemma 4.6.B (continued 1)

$$\begin{aligned} \text{Proof (continued). } P((a_1, b_1]) + P((a_2, b_2]) + \cdots + P((a_n, b_n]) \\ &= (F(b_1) - F(a_1)) + (F(b_2) - F(a_2)) + \cdots + (F(b_n) - F(a_n)) \\ &= -F(a_1) + (F(b_1) - F(a_2)) + (F(b_2) - F(a_3)) + \cdots \\ &\quad + (F(b_{n-1}) - F(a_n)) + F(b_n) \\ &= F(b_n) - F(a_1) \text{ since } a_i = b_{i-1} \text{ for } i = 2, 3, \dots, n \\ &= F(b) - F(a) \text{ since } a_1 = a \text{ and } b_n = b \\ &= P((a, b]) = P((a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n]). \end{aligned}$$

In Exercise 4.6.A, it is to be shown that  $P$  is countably monotone on  $\mathcal{S}$  (that is, if  $E = (a, b] \in \mathcal{S}$  and  $\{E_k\}_{k=1}^\infty$  is a countable collection of sets in  $\mathcal{S}$  such that  $E \subset \bigcup_{k=1}^\infty E_k \in \mathcal{S}$  then  $P(E) \leq \sum_{k=1}^\infty P(E_k)$ ). So  $P$  is a premeasure on semiring  $\mathcal{S}$ . By the Carathéodory-Hahn Theorem, there is a measure  $\bar{P}$  defined on the smallest  $\sigma$ -algebra containing  $\mathcal{S}$  that extends  $P$  and by the Carathéodory Theorem, since  $P$  is  $\sigma$ -finite,  $\bar{P}$  is unique. We also denote this measure as  $\bar{P} = P$ .

## Lemma 4.6.C

**Lemma 4.6.C.** Random variable  $X$  is continuous if and only if  $P(\{X = x\}) = 0$  for all  $x \in \mathbb{R}$ .

**Proof.** Let  $F$  be the distribution function of  $X$ :

$$F(x) = P(\{\omega \mid X(\omega) \leq x\}) = P(\{\omega \mid \omega \leq x\}) = P((-\infty, x]).$$

Then  $P((a, b]) = F(b) - F(a)$  from (\*). For  $n \in \mathbb{N}$ , consider  $\{(b - 1/n, b]\}_{n=1}^\infty$ . This is a decreasing sequence of sets with  $\lim_{n \rightarrow \infty} (b - 1/n, b] = \{b\}$ , so by Continuity of Measure  $P$  (Proposition 17.2(ii) of Royden and Fitzpatrick),

$$\begin{aligned} P(\{b\}) &= P\left(\lim_{n \rightarrow \infty} (b - 1/n, b]\right) = \lim_{n \rightarrow \infty} P((b - 1/n, b]) \\ &= \lim_{n \rightarrow \infty} F(b) - F(b - 1/n). \end{aligned}$$

## Lemma 4.6.C (continued)

**Lemma 4.6.C.** Random variable  $X$  is continuous if and only if  $P(\{X = x\}) = 0$  for all  $x \in \mathbb{R}$ .

**Proof (continued).** Now  $F$  is left-continuous on  $\mathbb{R}$  if and only if  $\lim_{n \rightarrow \infty} F(b - 1/n) = F(b)$  for all  $b \in \mathbb{R}$ ; that is, if and only if  $F$  is continuous at all  $b \in \mathbb{R}$  (since we already know by definition that  $F$  is right-continuous). So  $F$  is continuous if and only if  $P(\{b\}) = 0$  for all  $b \in \mathbb{R}$ . That is,  $X$  is continuous if and only if  $P(X = x) = 0$  for all  $x \in \mathbb{R}$ , as claimed.  $\square$