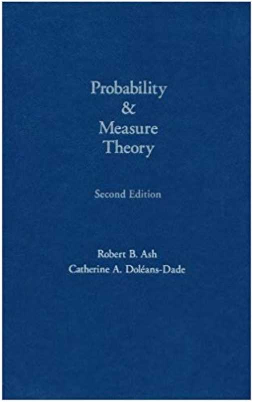
Real Analysis

Chapter 4. Basic Concepts of Probability

4.6. Random Variables—Proofs of Theorems



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Lemma 4.6.A

Lemma 4.6.A. Let X be a random variable on probability space $(\Omega, \mathcal{F}, P)$. Then the distribution function F of X is increasing and right-continuous. Also,

$$\lim_{x \to \infty} F(x) = 1 \text{ and } \lim_{x \to -\infty} F(x) = 0.$$

Proof. From $(*)$, for any $a < b$ we have

$$F(b) - F(a) = P_X((a, b]) = P(\{\omega \mid X(\omega) \in (a, b]\}) \geq 0$$

since P is a measure. So F is increasing, as claimed.

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$$\lim_{x \to b^+} F(x) = \lim_{h \to 0^+} F(b+h) = \lim_{h \to 0^+} P(\{\omega \mid X(\omega) \leq b+h\})$$

Since F is monotone increasing then one-sided limits exist, so we consider a sequence $\{b + 1/n\} \to b^+$ to evaluate $\lim_{x \to b^+} F(x)$:

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Lemma 4.6.A, continued 1

Proof (continued).

$$\begin{aligned}
\lim_{x\to b^+} F(x) &= \lim_{n\to\infty} P(\{\omega \mid X(\omega \le b + 1/n\}) \\
&= P\left(\lim_{n\to\infty}\{\omega \mid X(\omega) \le b + 1/n\}\right) \text{ by the} \\
&\quad \text{Continuity of Measure (Proposition 17.2),} \\
&\quad \text{since } \{E_n\}_{n=1}^\infty \text{ where } E_n = \{\omega \mid X(\omega) \le b + 1/n\} \\
&\quad \text{is a descending sequence of sets} \\
&= P\left(\cap_{n=1}^\infty \{\omega \mid X(\omega) \le b + 1/n\}\right) \\
&= P(\{\omega \mid X(\omega) \le b\}) = F(b).
\end{aligned}$$

So F is right-continuous. Since P is a measure, $P(\varnothing) = 0$ and since $(\Omega, \mathcal{F}, P)$ is a probability space then $P(\Omega) = 1$.

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$$\begin{aligned}\lim_{x\to b^+} F(x) &= \lim_{n\to\infty} P(\{\omega \mid X(\omega \le b+1/n\}) \\ &= P\left(\lim_{n\to\infty}\{\omega \mid X(\omega) \le b+1/n\}\right) \text{ by the} \\ &\quad \text{Continuity of Measure (Proposition 17.2),} \\ &\quad \text{since } \{E_n\}_{n=1}^{\infty} \text{ where } E_n = \{\omega \mid X(\omega) \le b+1/n\} \\ &\quad \text{is a descending sequence of sets} \\ &= P\left(\cap_{n=1}^{\infty}\{\omega \mid X(\omega) \le b+1/n\}\right) \\ &= P(\{\omega \mid X(\omega) \le b\}) = F(b).\end{aligned}$$

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Lemma 4.6.A, continued 2

Proof (continued). Again by Continuity of Measure

$$\begin{aligned} 0 &= P(\varnothing) = P\left(\cap_{n=1}^{\infty}\{\omega \mid X(\omega) \le -n\}\right) = P\left(\lim_{n\to\infty}\{\omega \mid X(\omega) \le -n\}\right) \\ &= \lim_{n\to\infty} P\left(\{\omega \mid X(\omega) \le -n\}\right) = \lim_{n\to\infty} F(-n) \\ &= \lim_{n\to-\infty} F(x) \text{ since } F \text{ is monotone increasing,} \end{aligned}$$

and

$$\begin{aligned} 1 &= P(\Omega) = P\left(\cup_{n=1}^{\infty}\{\omega \mid X(\omega) \ge n\}\right) \\ &= P\left(\lim_{n\to\infty}\{\omega \mid X(\omega) \ge n\}\right) \\ &= \lim_{n\to\infty} P(\{\omega \mid X(\omega) \ge n\}) = \lim_{n\to\infty} F(n) = \lim_{x\to\infty} F(x), \end{aligned}$$

as claimed. □

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Lemma 4.6.B

Lemma 4.6.B. If $F : \mathbb{R} \to [0,1]$ is an increasing and right-continuous function with $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$, then F is the distribution function of some random variable. Note: Though F is defined on $\mathbb{R}$, we denote $\lim_{x\to\infty} F(x) = F(\infty)$ and $\lim_{x\to-\infty} F(x) = F(-\infty)$.

Proof. We take $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$ (the Borel sets in $\mathbb{R}$), and define $X(\omega) = \omega$ for $\omega \in \Omega$ (s X is the identity map on $\Omega = \mathbb{R}$). We use F to define a probability measure P; define $P((a,b]) = F(b) = F(a)$ for all $a, b \in \mathbb{R}$ and define $P(\varnothing) = 0$. Then P is defined on all $(a,b] \subset \mathbb{R}$ and the set of all such $(a,b]$ (along with $\varnothing$) form a semiring $\mathcal{S}$.

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$$(a, b] = (a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n]$$

(where, say, $a_1 = a$, $b_n = b$, and $a_i = b_{i-1}$ for $i = 2, 3, \ldots, n$) then...

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Lemma 4.6.B (continued 1)

Proof (continued). $P((a_1, b_1]) + P((a_2, b_2]) + \cdots P((a_n, b_n])$

$$\begin{aligned} &= (F(b_1) - F(a_1)) + (F(b_2) - F(a_2)) + \cdots + (F(b_n) - F(a_n)) \\ &= -F(a_1) + (F(b_1) - F(a_2)) + (F(b_2) - F(a_3)) + \cdots \\ &\quad +(F(b_{n-1}) - F(a_n)) + F(b_n) \\ &= F(b_n) - F(a_1) \text{ since } a_i = b_{i-1} \text{ for } i = 2, 3, \ldots, n \\ &= F(b) - F(a) \text{ since } a_1 = a \text{ and } b_n = b \\ &= P((a, b]) = P((a_1, b_1] \mathbin{\dot\cup} (a_2, b_2] \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} (a_n, b_n]). \end{aligned}$$

In Exercise 4.6.A, it is to be shown that P is countably monotone on $\mathcal{S}$ (that is, if $E = (a, b] \in \mathcal{S}$ and $\{E_k\}_{k=1}^{\infty}$ is a countable collection of sets in $\mathcal{S}$ such that $E \subset \cup_{k=1}^{\infty} E_k \in \mathcal{S}$ then $P(E) \leq \sum_{k=1}^{\infty} P(E_k)$). So P is a premeasure on semiring $\mathcal{S}$. By the Carathéodory-Hahn Theorem, there is a measure $\overline{P}$ defined on the smallest σ-algebra containing $\mathcal{S}$ that extends P and by the Carathéodory Theorem, since P is σ-finite, $\overline{P}$ is unique. We also denote this measure as $\overline{P} = P$.

Lemma 4.6.B (continued 1)

Proof (continued). $P((a_1, b_1]) + P((a_2, b_2]) + \cdots P((a_n, b_n])$

$$\begin{aligned}
&= (F(b_1) - F(a_1)) + (F(b_2) - F(a_2)) + \cdots + (F(b_n) - F(a_n)) \\
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&= F(b_n) - F(a_1) \text{ since } a_i = b_{i-1} \text{ for } i = 2, 3, \ldots, n \\
&= F(b) - F(a) \text{ since } a_1 = a \text{ and } b_n = b \\
&= P((a, b]) = P((a_1, b_1] \,\dot\cup\, (a_2, b_2] \,\dot\cup \cdots \dot\cup\, (a_n, b_n]).
\end{aligned}$$

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Lemma 4.6.B (continued 2)

Proof (continued). (Notice that the smallest σ-algebra containing all subsets of $\mathbb{R}$ of the form $(a,b]$ is $\mathcal{B}(\mathbb{R})$, by Exercise 1.36 in Royden and Fitzpatrick.) So $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P)$ is a probability space. For $x \in \mathbb{R}$,

$$\begin{aligned} F(x) &= F(x) - F(-\infty) \text{ since } F(-\infty) = \lim_{y\to-\infty} F(y) = 0 \\ &= F(x) - \lim_{y\to-\infty} F(y) = \lim_{y\to-\infty} (F(x) - F(y)) \\ &= \lim_{y\to\infty} P((y,x]) = \lim_{n\to-\infty} P((-n,x]) \text{ since } P \text{ is monotone} \\ &\quad \text{and } P((y,x]) \text{ is a decreasing function of } y \text{ so the limit can} \\ &\quad \text{be evaluated using any sequence of values } \{y_n\} \to -\infty \\ &= P\left(\lim_{n\to-\infty} (-n,x]\right) \text{ by Continuity of Measure since } \{(-n,x]\} \\ &\quad \text{is an increasing sequence of sets (for } n \text{ sufficiently large)} \\ &= P((-\infty,x]) = P(\omega \mid \omega \le x) = P(\omega \mid X(\omega) \le x\} \text{ since } X(\omega) = \omega, \end{aligned}$$

so F is the distribution of random variable X, as claimed. □

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&= P\left(\lim_{n\to-\infty}(-n, x]\right) \text{ by Continuity of Measure since } \{(-n, x]\} \\
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\end{aligned}$$

so F is the distribution of random variable X, as claimed. □

Lemma 4.6.C

Lemma 4.6.C. Random variable X is continuous if and only if $P(\{X = x\}) = 0$ for all $x \in \mathbb{R}$.

Proof. Let F be the distribution function of X:

$$F(x) = P(\{\omega \mid X(\omega) \leq x\}) = P(\{\omega \mid \omega \leq x\}) = P((-\infty, x]).$$

Then $P((a, b]) = F(b) - F(a)$ from $(*)$. For $n \in \mathbb{N}$, consider $\{(b - 1/n, b]\}_{n=1}^{\infty}$.

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$$P(\{b\}) = P\left(\lim_{n\to\infty}(b - 1/n, b]\right) = \lim_{n\to\infty} P((b - 1/n, b])$$

$$= \lim_{n\to\infty} F(b) - F(b - 1/n).$$

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$$= \lim_{n\to\infty} F(b) - F(b - 1/n).$$

Lemma 4.6.C (continued)

Lemma 4.6.C. Random variable X is continuous if and only if $P(\{X = x\}) = 0$ for all $x \in \mathbb{R}$.

Proof (continued). Now F is left-continuous on $\mathbb{R}$ if and only if $\lim_{n\to\infty} F(b - 1/n) = F(b)$ for all $b \in \mathbb{R}$; that is, if and only if F is continuous at all $b \in \mathbb{R}$ (since we already know by definition that F is right-continuous). So F is continuous if and only if $P(\{b\}) = 0$ for all $b \in \mathbb{R}$. That is, X is continuous if and only if $P(X = x) = 0$ for all $x \in \mathbb{R}$, as claimed. □