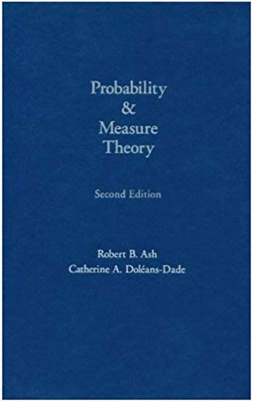
Real Analysis

Chapter 4. Basic Concepts of Probability

4.8. Independent Random Variables—Proofs of Theorems



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Proof. Let $\mathcal{F} = \sigma(\mathcal{F}_0)$ and let $\mathcal{M}$ be the smallest monotone class containing all sets of $\mathcal{F}_0$. We will show that $\mathcal{M} = \mathcal{F}$; that is, the smallest monotone class containing field $\mathcal{F}_0$ and the smallest σ-field over field $\mathcal{F}_0$ are the same.

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Let $A \in \mathcal{M}$ be fixed. Define

$$\mathcal{M}_A = \{B \in \mathcal{M} \mid A \cap B, A \cap B^c, A^c \cap B \in \mathcal{M}\} \subset \mathcal{M}.$$

If $\{B_n\}_{n=1}^{\infty} \subset \mathcal{M}_A$ with $B_1 \subset B_2 \subset \cdots$ then $A \cap B_n, A \cap B_n^c, A^c \cap B_n^c \in \mathcal{M}$ for all $n \in \mathbb{N}$.

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If $\{B_n\}_{n=1}^{\infty} \subset \mathcal{M}_A$ with $B_1 \subset B_2 \subset \cdots$ then $A \cap B_n, A \cap B_n^c, A^c \cap B_n^c \in \mathcal{M}$ for all $n \in \mathbb{N}$. Then $\cup_{n=1}^{\infty} B_n \in \mathcal{M}$ since $\mathcal{M}$ is monotone, and $A \cap (\cup_{n=1}^{\infty} B_n) = \cup_{n=1}^{\infty}(A \cap B_n) \in \mathcal{M}$ since $(A \cap B_1) \subset (A \cap B_2) \subset \cdots$ and $\mathcal{M}$ is monotone. Similarly, $A^c \cap (\cup_{n=1}^{\infty} B_n) = \cup_{n=1}^{\infty}(A^c \cap B_n) \in \mathcal{M}$.

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If $\{B_n\}_{n=1}^{\infty} \subset \mathcal{M}_A$ with $B_1 \subset B_2 \subset \cdots$ then $A \cap B_n, A \cap B_n^c, A^c \cap B_n^c \in \mathcal{M}$ for all $n \in \mathbb{N}$. Then $\cup_{n=1}^{\infty} B_n \in \mathcal{M}$ since $\mathcal{M}$ is monotone, and $A \cap (\cup_{n=1}^{\infty} B_n) = \cup_{n=1}^{\infty}(A \cap B_n) \in \mathcal{M}$ since $(A \cap B_1) \subset (A \cap B_2) \subset \cdots$ and $\mathcal{M}$ is monotone. Similarly, $A^c \cap (\cup_{n=1}^{\infty} B_n) = \cup_{n=1}^{\infty}(A^c \cap B_n) \in \mathcal{M}$.

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Theorem 1.3.9. The Monotone Class Theorem.
Let $\mathcal{F}_0$ be a field of subsets of Ω and $\mathcal{C}$ a class of subsets of Ω that is monotone. If $\mathcal{C} \supset \mathcal{F}_0$, then $\mathcal{C} \supset \sigma(\mathcal{F}_0)$, the minimal σ-filed over $\mathcal{F}_0$.

Proof (continued). Also, $(\cup_{n=1}^\infty B_n)^c = \cap_{n=1}^\infty B_n^c \in \mathcal{M}$ since $B_1^c \supset B_2^c \supset \cdots$ and $\mathcal{M}$ is monotone. Next, $A \cap (\cup_{n=1}^\infty B_n)^c = A \cap (\cap_{n=1}^\infty B_n^c) = \cap_{n=1}^\infty (A \cap B_n^c) \in \mathcal{M}$ since $(A \cap B_1^c) \supset (A \cap B_2^c) \supset \cdots$ and $\mathcal{M}$ is monotone.

If $\{B_1\}_{n=1}^\infty \subset \mathcal{M}_A$ with $B_1 \supset B_2 \supset \cdots$ then $A \cap B_n, A \cap B_n^c, A^c \cap B_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. Then $\cap_{n=1}^\infty B_n \in \mathcal{M}$ since $\mathcal{M}$ is monotone, and $A \cap (\cap_{n=1}^\infty B_n) = \cap_{n=1}^\infty (A \cap B_n) \in \mathcal{M}$ since $(A \cap B_1) \supset (A \cap B_2) \supset \cdots$ and $\mathcal{M}$ is monotone. Similarly, $A^c \cap (\cap_{n=1}^\infty B_n) = \cap_{n=1}^\infty (A^c \cap B_n) \in \mathcal{M}$.

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Proof (continued). Also, $(\cup_{n=1}^{\infty} B_n)^c = \cap_{n=1}^{\infty} B_n^c \in \mathcal{M}$ since $B_1^c \supset B_2^c \supset \cdots$ and $\mathcal{M}$ is monotone. Next, $A \cap (\cup_{n=1}^{\infty} B_n)^c = A \cap (\cap_{n=1}^{\infty} B_n^c) = \cap_{n=1}^{\infty}(A \cap B_n^c) \in \mathcal{M}$ since $(A \cap B_1^c) \supset (A \cap B_2^c) \supset \cdots$ and $\mathcal{M}$ is monotone.

If $\{B_1\}_{n=1}^{\infty} \subset \mathcal{M}_A$ with $B_1 \supset B_2 \supset \cdots$ then $A \cap B_n, A \cap B_n^c, A^c \cap B_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. Then $\cap_{n=1}^{\infty} B_n \in \mathcal{M}$ since $\mathcal{M}$ is monotone, and $A \cap (\cap_{n=1}^{\infty} B_n) = \cap_{n=1}^{\infty}(A \cap B_n) \in \mathcal{M}$ since $(A \cap B_1) \supset (A \cap B_2) \supset \cdots$ and $\mathcal{M}$ is monotone. Similarly, $A^c \cap (\cap_{n=1}^{\infty} B_n) = \cap_{n=1}^{\infty}(A^c \cap B_n) \in \mathcal{M}$. Also, $(cap_{n=1}^{\infty} B_n)^c = \cup_{n=1}^{\infty} B_n^c \in \mathcal{M}$ since $B_1^c \subset B_2^c \subset \cdots$ and $\mathcal{M}$ is monotone. Next, $A \cap (\cap_{n=1}^{\infty} B_n)^c = A \cap (\cup_{n=1}^{\infty} B_n^c) = \cup_{n=1}^{\infty}(A \cap B_n^c) \in \mathcal{M}$ since $(A \cap B_1^c) \subset (A \cap B_2^c) \subset \cdots$ and $\mathcal{M}$ is monotone. Therefore, $\mathcal{M}_A$ is a monotone class.

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Proof (continued). Also, $(\cup_{n=1}^{\infty} B_n)^c = \cap_{n=1}^{\infty} B_n^c \in \mathcal{M}$ since $B_1^c \supset B_2^c \supset \cdots$ and $\mathcal{M}$ is monotone. Next, $A \cap (\cup_{n=1}^{\infty} B_n)^c = A \cap (\cap_{n=1}^{\infty} B_n^c) = \cap_{n=1}^{\infty}(A \cap B_n^c) \in \mathcal{M}$ since $(A \cap B_1^c) \supset (A \cap B_2^c) \supset \cdots$ and $\mathcal{M}$ is monotone.

If $\{B_1\}_{n=1}^{\infty} \subset \mathcal{M}_A$ with $B_1 \supset B_2 \supset \cdots$ then $A \cap B_n, A \cap B_n^c, A^c \cap B_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. Then $\cap_{n=1}^{\infty} B_n \in \mathcal{M}$ since $\mathcal{M}$ is monotone, and $A \cap (\cap_{n=1}^{\infty} B_n) = \cap_{n=1}^{\infty}(A \cap B_n) \in \mathcal{M}$ since $(A \cap B_1) \supset (A \cap B_2) \supset \cdots$ and $\mathcal{M}$ is monotone. Similarly, $A^c \cap (\cap_{n=1}^{\infty} B_n) = \cap_{n=1}^{\infty}(A^c \cap B_n) \in \mathcal{M}$. Also, $(cap_{n=1}^{\infty} B_n)^c = \cup_{n=1}^{\infty} B_n^c \in \mathcal{M}$ since $B_1^c \subset B_2^c \subset \cdots$ and $\mathcal{M}$ is monotone. Next, $A \cap (\cap_{n=1}^{\infty} B_n)^c = A \cap (\cup_{n=1}^{\infty} B_n^c) = \cup_{n=1}^{\infty}(A \cap B_n^c) \in \mathcal{M}$ since $(A \cap B_1^c) \subset (A \cap B_2^c) \subset \cdots$ and $\mathcal{M}$ is monotone. Therefore, $\mathcal{M}_A$ is a monotone class.

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Theorem 1.3.9. The Monotone Class Theorem.
Let $\mathcal{F}_0$ be a field of subsets of Ω and $\mathcal{C}$ a class of subsets of Ω that is monotone. If $\mathcal{C} \supset \mathcal{F}_0$, then $\mathcal{C} \supset \sigma(\mathcal{F}_0)$, the minimal σ-filed over $\mathcal{F}_0$.

Proof (continued). If $A \in \mathcal{F}_0$ then $A^c \in \mathcal{F}_0$ since $\mathcal{F}_0$ is a field, and for any $B \in \mathcal{F}_0$ we must also have $A \cap B, B^c, A \cap B^c, A^c \cap B \in \mathcal{F}_0$ since $\mathcal{F}_0$ is a field. Since B satisfies these conditions, then $B \in \mathcal{M}_A$; that is, if $A \in \mathcal{F}_0$ then $\mathcal{F}_0 \subset \mathcal{M}_A$. Since $\mathcal{M}$ is the smallest monotone monotone class containing all sets in $\mathcal{F}_0$ and $\mathcal{M}_A$ is a monotone class containing $\mathcal{F}_0$ (if $A \in \mathcal{F}_0$) then we must have $\mathcal{M} \subset \mathcal{M}_A$. Since $\mathcal{M}_A \subset \mathcal{M}$ by the definition of $\mathcal{M}_!$, then we have $\mathcal{M}_A = \mathcal{M}$ for any $A \in \mathcal{F}_0$. So for any $B \in \mathcal{M} = \mathcal{M}_A$ we have $A \cap B, A \cap B^c, A^c \cap B \in \mathcal{M} = \mathcal{M}_A$, provided $A \in \mathcal{F}_0$. That is, $A \in \mathcal{F}_0$ implies $A \in \mathcal{M}_B$, so that $\mathcal{F}_0 \subset \mathcal{M}_B$. But as just shown for $\mathcal{M}_A$, $\mathcal{M}_B$ is also a monotone class (regardless of whether $B \in \mathcal{F}_0$ or not), and so $\mathcal{M}_B$ is a monotone class containing $\mathcal{F}_0$ so that $\mathcal{M}_B \supset \mathcal{M}$ by the minimality definition of $\mathcal{M}$. But $\mathcal{M}_B \subset \mathcal{M}$ by the definition of $\mathcal{M}_B$, so that $\mathcal{M}_B = \mathcal{M}$ for any $B \in \mathcal{M}$.

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Theorem 1.3.9 (continued 3)

Proof (continued). Next, for any $A, B \in \mathcal{M} = \mathcal{M}_A$, we have (by the definition of $\mathcal{M}_A$) $A \cap B, A \cap B^c, A^c \cap B \in \mathcal{M} = \mathcal{M}_A$ so that $\mathcal{M}$ is closed under complements (since by definition field $\Omega \in \mathcal{F}_0$ so for $A \in \mathcal{M}$, $A^c \cap \Omega = A^c \in \mathcal{M}$) and finite intersections. That is, $\mathcal{M}$ is a field. We also claim that $\mathcal{M}$ is a σ-field. Let $A_1, A_2, \ldots \in \mathcal{M}$. Then $A_1 \subset (A_1 \cup A_2) \subset (A_1 \cup A_2 \cup A_3) \subset$ and since $\mathcal{M}$ is a monotone class, $\cup_{n=1}^\infty A_n = \cup_{k=1}^\infty (A_1 \cup A_2 \cup \cdots \cup A_k) \in \mathcal{M}$. Also, $A_1 \supset (A_1 \cup A_2) \supset (A_1 \cap A_2 \cap A_3) \supset \cdots$ and since $\mathcal{M}$ is a monotone class $\cap_{n=1}^\infty A_n = \cap_{k=1}^\infty (A_1 \cap A_2 \cap \cdots \cap A_k) \in \mathcal{M}$. That is, $\mathcal{M}$ is a field closed under countable unions (and countable intersections). So $\mathcal{M}$ is a σ-field which contains $\mathcal{F}_0$.

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Theorem 1.3.9 (continued 3)

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Since $\mathcal{C}$ is a monotone class of subsets of Ω containing $\mathcal{F}_0$, then $\mathcal{C} \supset \mathcal{M} \supset \mathcal{F}_0$, and so we have $\mathcal{C} \supset \mathcal{M} = \mathcal{F} = \sigma(\mathcal{F}_0)$, as claimed. □

Theorem 1.3.9 (continued 3)

Proof (continued). Next, for any $A, B \in \mathcal{M} = \mathcal{M}_A$, we have (by the definition of $\mathcal{M}_A$) $A \cap B, A \cap B^c, A^c \cap B \in \mathcal{M} = \mathcal{M}_A$ so that $\mathcal{M}$ is closed under complements (since by definition field $\Omega \in \mathcal{F}_0$ so for $A \in \mathcal{M}$, $A^c \cap \Omega = A^c \in \mathcal{M}$) and finite intersections. That is, $\mathcal{M}$ is a field. We also claim that $\mathcal{M}$ is a σ-field. Let $A_1, A_2, \ldots \in \mathcal{M}$. Then $A_1 \subset (A_1 \cup A_2) \subset (A_1 \cup A_2 \cup A_3) \subset$ and since $\mathcal{M}$ is a monotone class, $\cup_{n=1}^{\infty} A_n = \cup_{k=1}^{\infty}(A_1 \cup A_2 \cup \cdots \cup A_k) \in \mathcal{M}$. Also, $A_1 \supset (A_1 \cup A_2) \supset (A_1 \cap A_2 \cap A_3) \supset \cdots$ and since $\mathcal{M}$ is a monotone class $\cap_{n=1}^{\infty} A_n = \cap_{k=1}^{\infty}(A_1 \cap A_2 \cap \cdots \cap A_k) \in \mathcal{M}$. That is, $\mathcal{M}$ is a field closed under countable unions (and countable intersections). So $\mathcal{M}$ is a σ-field which contains $\mathcal{F}_0$. Since $\mathcal{F} = \sigma(\mathcal{F}_0)$ is the minimal σ-field containing $\mathcal{F}_0$, then $\mathcal{F} \subset \mathcal{M}$. Since $\mathcal{F}$ is a σ-field, then it is also a monotone class (containing $\mathcal{F}_0$), and by the minimality of monotone class $\mathcal{M}$, we have $\mathcal{M} \subset \mathcal{F}$. Therefore $\mathcal{M} = \mathcal{F}$.

Since $\mathcal{C}$ is a monotone class of subsets of Ω containing $\mathcal{F}_0$, then $\mathcal{C} \supset \mathcal{M} \supset \mathcal{F}_0$, and so we have $\mathcal{C} \supset \mathcal{M} = \mathcal{F} = \sigma(\mathcal{F}_0)$, as claimed. □

Theorem 4.8.3

Theorem 4.8.3. Let $X_1, X_2, \ldots, X_n$ be random variables on $(\Omega, \mathcal{F}, P)$. Let F_i by the distribution function of X_i, $i = 1, 2, \ldots, n$ (so $F_i(x_i) = P(\{X_i \leq x_i\})$) and F the distribution function of $X = (X_1, X_2, \ldots, X_n)$ (that is, $F(x_1, x_2, \ldots, x_n) = P(\{X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n\})$). Then $X_1, X_2, \ldots, X_n$ are independent if and only if $F(x_1, x_2, \ldots, x_n) = F_1(x_1)F_2(x_2) \cdots F_n(x_n)$ for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

Proof. If $X_1, X_2, \ldots, X_n$ are independent then

$$F(x_1, x_2, \ldots, x_n) = P(\{X \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n\})$$

$$= P(\{X_1 \leq x_1\})P(\{X_2 \leq x_2\}) \cdots P(\{X_n \leq x_n\}) = F_1(x_1)F_2(x_2) \cdots F_n(x_n),$$

as claimed.

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Proof. If $X_1, X_2, \ldots, X_n$ are independent then

$$F(x_1, x_2, \ldots, x_n) = P(\{X \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n\})$$

$$= P(\{X_1 \leq x_1\})P(\{X_2 \leq x_2\})\cdots P(\{X_n \leq x_n\}) = F_1(x_1)F_2(x_2)\cdots F_n(x_n),$$

as claimed.

Theorem 4.8.3 (continued 1)

Proof (continued). Now suppose

$F(x_1, x_2, \ldots, x_n) = F_1(x_1)F_2(x_2)\cdots F_n(x_n)$ for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Let $a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ with $a \le b$ (that is, $a_i \le b_i$ for $i = 1, 2, \ldots, n$). Then

$$(a, b] = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid a_i < x_i \le b_1 \text{ for } i = 1, 2, \ldots, n\}$$

(see page 26 of the text). So

$$\begin{aligned} P_X((a,b]) &= P(\{X \in (a,b]\}) \\ &= P(\{a_1 < X_1 \le b_1, a_2 < X_2 \le b_2, \ldots, a_n < X_n \le b_n\}) \\ &= F((a,b]) = (F_1(b_1) - F_1(a_1))(F_2(b_2) - F_2(a_2))\cdots \\ &\quad (F_n(b_n) - F_n(a_n)) \text{by Example 1.4.10(a) of the text} \\ &\quad \text{ and the hypothesis on } F \\ &= P_{X_1}(a_1, b_1])P_{X_2}((a_2, b_2])\cdots P_{X_n}((a_n, b_n]). \end{aligned}$$

Theorem 4.8.3 (continued 1)

Proof (continued). Now suppose

$F(x_1, x_2, \ldots, x_n) = F_1(x_1)F_2(x_2)\cdots F_n(x_n)$ for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Let $a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ with $a \leq b$ (that is, $a_i \leq b_i$ for $i = 1, 2, \ldots, n$). Then

$$(a, b] = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid a_i < x_i \leq b_1 \text{ for } i = 1, 2, \ldots, n\}$$

(see page 26 of the text). So

$$\begin{aligned} P_X((a, b]) &= P(\{X \in (a, b]\}) \\ &= P(\{a_1 < X_1 \leq b_1, a_2 < X_2 \leq b_2, \ldots, a_n < X_n \leq b_n\}) \\ &= F((a, b]) = (F_1(b_1) - F_1(a_1))(F_2(b_2) - F_2(a_2))\cdots \\ &\quad (F_n(b_n) - F_n(a_n)) \text{by Example 1.4.10(a) of the text} \\ &\quad \text{ and the hypothesis on } F \\ &= P_{X_1}(a_1, b_1])P_{X_2}((a_2, b_2])\cdots P_{X_n}((a_n, b_n]). \end{aligned}$$

Theorem 4.8.3 (continued 2)

Proof (continued). So

$$P(\{X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n\})$$
$$= P(\{X_1 \in B_1\})P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\}) \quad (1)$$

holds when each B_i is of the form $B_i = (a_i, b_i] \subset \mathbb{R}$.

Now fix the intervals $B_2, B_3, \ldots, B_n$ and let $\mathcal{C}$ be the collection of sets $B_1 \in \mathcal{B}(\mathbb{R})$ for which equation (1) holds (so B_1 need not be of the form $(a_1, b_1]$ here, but $\mathcal{C}$ does include all intervals of this form). Suppose $\{A_n\}_{n=1}^{\infty} \subset \mathcal{C}$ and $A_1 \subset A_2 \subset \cdots$.

Theorem 4.8.3 (continued 2)

Proof (continued). So

$$P(\{X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n\})$$
$$= P(\{X_1 \in B_1\})P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\}) \quad (1)$$

holds when each B_i is of the form $B_i = (a_i, b_i] \subset \mathbb{R}$.

Now fix the intervals $B_2, B_3, \ldots, B_n$ and let $\mathcal{C}$ be the collection of sets $B_1 \in \mathcal{B}(\mathbb{R})$ for which equation (1) holds (so B_1 need not be of the form $(a_1, b_1]$ here, but $\mathcal{C}$ does include all intervals of this form). Suppose $\{A_n\}_{n=1}^{\infty} \subset \mathcal{C}$ and $A_1 \subset A_2 \subset \cdots$. Then

$$\{X_1 \in \cup_{m=1}^{k} A_m, X_2 \in B_2, \ldots, X_n \in B_n\} = \{X_1 \in A_k, X_2 \in B_2, \ldots, X_n \in B_n\}$$

and since $A_k \in \mathcal{C}$ then

$$P\left(\left\{X_1 \in \cup_{m=1}^{k} A_m, X_2 \in B_2, \ldots, X_n \in B_n\right\}\right)$$
$$= P(\{X_1 \in A_k, X_2 \in B_2, \ldots, X_n \in B_n\})$$

Theorem 4.8.3 (continued 2)

Proof (continued). So

$$P(\{X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n\})$$
$$= P(\{X_1 \in B_1\})P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\}) \quad (1)$$

holds when each B_i is of the form $B_i = (a_i, b_i] \subset \mathbb{R}$.

Now fix the intervals $B_2, B_3, \ldots, B_n$ and let $\mathcal{C}$ be the collection of sets $B_1 \in \mathcal{B}(\mathbb{R})$ for which equation (1) holds (so B_1 need not be of the form $(a_1, b_1]$ here, but $\mathcal{C}$ does include all intervals of this form). Suppose $\{A_n\}_{n=1}^{\infty} \subset \mathcal{C}$ and $A_1 \subset A_2 \subset \cdots$. Then

$$\{X_1 \in \cup_{m=1}^{k} A_m, X_2 \in B_2, \ldots, X_n \in B_n\} = \{X_1 \in A_k, X_2 \in B_2, \ldots, X_n \in B_n\}$$

and since $A_k \in \mathcal{C}$ then

$$P\left(\left\{X_1 \in \cup_{m=1}^{k} A_m, X_2 \in B_2, \ldots, X_n \in B_n\right\}\right)$$
$$= P(\{X_1 \in A_k, X_2 \in B_2, \ldots, X_n \in B_n\})$$

Theorem 4.8.3 (continued 3)

Proof (continued). ...

$$= P(\{X_1 \in A_k\})P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\})$$
$$= P\left(\left\{X_1 \in \cup_{m=1}^{k} A_m\right\}\right) P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\}).$$

Now

$$\left\{X_1 \in \cup_{m=1}^{k} A_m, X_2 \in B_2, \ldots, X_n \in B_n\right\}$$
$$\subset \left\{X_1 \in \cup_{m=1}^{k+1} A_m, X_2 \in B_2, \ldots, X_n \in B_n\right\}$$

so this sequence of sets of events indexed by k is ascending so that

$$\lim_{k\to\infty} \left\{X_1 \in \cup_{m=1}^{k} A_m, X_2 \in B_2, \ldots, X_n \in B_n\right\}$$
$$= \left\{X_1 \in \cup_{m=1}^{k} A_m, X_2 \in B_2, \ldots, X_n \in B_n\right\}$$

and since probability measure P is continuous (Proposition 17.2(i) of Royden and Fitzpatrick)...

Theorem 4.8.3 (continued 3)

Proof (continued). ...

$$= P(\{X_1 \in A_k\})P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\})$$
$$= P\left(\left\{X_1 \in \cup_{m=1}^{k} A_m\right\}\right) P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\}).$$

Now

$$\left\{X_1 \in \cup_{m=1}^{k} A_m, X_2 \in B_2, \ldots, X_n \in B_n\right\}$$
$$\subset \left\{X_1 \in \cup_{m=1}^{k+1} A_m, X_2 \in B_2, \ldots, X_n \in B_n\right\}$$

so this sequence of sets of events indexed by k is ascending so that

$$\lim_{k \to \infty} \left\{X_1 \in \cup_{m=1}^{k} A_m, X_2 \in B_2, \ldots, X_n \in B_n\right\}$$
$$= \left\{X_1 \in \cup_{m=1}^{k} A_m, X_2 \in B_2, \ldots, X_n \in B_n\right\}$$

and since probability measure P is continuous (Proposition 17.2(i) of Royden and Fitzpatrick)...

Theorem 4.8.3 (continued 4)

Proof (continued). ...

$$P\left(\{X_1 \in \cup_{m=1}^{\infty} A_m, X_2 \in B_2, \ldots, X_n \in B_n\}\right)$$

$$\begin{aligned}
&= P\left(\lim_{k\to\infty}\left\{X_1 \in \cup_{m=1}^{k} A_m, X_2 \in B_2, \ldots, X_n \in B_n\right\}\right) \\
&= P\left(\lim_{k\to\infty}\{X_1 \in A_k, X_2 \in B_2, \ldots, X_n \in B_n\}\right) \\
&= \lim_{k\to\infty} P\left(\{X_1 \in A_k, X_2 \in B_2, \ldots, X_n \in B_n\}\right) \\
&= \lim_{k\to\infty} P(\{X_1 \in A_k\})P(\{X_2 \in B_2\}\cdots P(\{X_n \in B_n\}) \text{ since } A_k \in \mathcal{C} \\
&= P\left(\lim_{k\to\infty}\{X_1 \in A_k\}\right) P(\{X_2 \in B_2\}\cdots P(\{X_n \in B_n\} \\
&= P\left(\{X_1 \in \cup_{m=1}^{\infty} A_m\}\right) P(\{X_2 \in B_2\}\cdots P(\{X_n \in B_n\})
\end{aligned}$$

and so $\cup_{m=1}^{\infty} A_m \in \mathcal{C}$.

Theorem 4.8.3 (continued 5)

Proof (continued). Similarly, if $\{A_m\}_{m=1}^{\infty} \subset \mathcal{C}$ with $A_1 \supset A_2 \supset$, we have by the continuity of probability measure P (Proposition 17.2(ii) of Royden and Fitzpatrick) that $\cap_{m=1}^{\infty} A_m \in \mathcal{C}$. So collection $\mathcal{C}$ for which (1) holds is a monotone class. Now if B_1 is a finite union of disjoint "right-semiclosed intervals," $B_1 = (a_1^1, b_1^1] \cup (a_1^2, b_1^2] \cup \cdots \cup (a_1^\ell, b_1^\ell]$ then

$$\begin{aligned}\{X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n\} &= \{X_1 \in (a_1^1, b_1^1], X_2 \in B_2, \ldots X_n \in B_n\} \\ &\cup \{X_1 \in (a_1^2, b_1^2], X_2 \in B_2, \ldots X_n \in B_n\} \cup \cdots \\ &\cup \{X_1 \in (a_1^\ell, b_1^\ell], X_2 \in B_2, \ldots X_n \in B_n\},\end{aligned}$$

so by finite additivity of probability measure P,

$$\begin{aligned}& P(\{X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n\}) \\ = \; & P(\{X_1 \in (a_1^1, b_1^1], X_2 \in B_2, \cdots, X_n \in B_n\}) \\ & + P(\{X_1 \in (a_1^2, b_1^2], X_2 \in B_2, \ldots X_n \in B_n\}) + \cdots \\ & + P(\{X_1 \in (a_1^\ell, b_1^\ell], X_2 \in B_2, \ldots X_n \in B_n\})\end{aligned}$$

Theorem 4.8.3 (continued 5)

Proof (continued). Similarly, if $\{A_m\}_{m=1}^{\infty} \subset \mathcal{C}$ with $A_1 \supset A_2 \supset$, we have by the continuity of probability measure P (Proposition 17.2(ii) of Royden and Fitzpatrick) that $\cap_{m=1}^{\infty} A_m \in \mathcal{C}$. So collection $\mathcal{C}$ for which (1) holds is a monotone class. Now if B_1 is a finite union of disjoint "right-semiclosed intervals," $B_1 = (a_1^1, b_1^1] \cupdot (a_1^2, b_1^2] \cupdot \cdots \cupdot (a_1^\ell, b_1^\ell]$ then

$$\begin{aligned}\{X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n\} &= \{X_1 \in (a_1^1, b_1^1], X_2 \in B_2, \ldots X_n \in B_n\} \\ &\cupdot \{X_1 \in (a_1^2, b_1^2], X_2 \in B_2, \ldots X_n \in B_n\} \cupdot \cdots \\ &\cupdot \{X_1 \in (a_1^\ell, b_1^\ell], X_2 \in B_2, \ldots X_n \in B_n\},\end{aligned}$$

so by finite additivity of probability measure P,

$$\begin{aligned}& P(\{X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n\}) \\ = \; & P(\{X_1 \in (a_1^1, b_1^1], X_2 \in B_2, \cdots, X_n \in B_n\}) \\ & + P(\{X_1 \in (a_1^2, b_1^2], X_2 \in B_2, \ldots X_n \in B_n\}) + \cdots \\ & + P(\{X_1 \in (a_1^\ell, b_1^\ell], X_2 \in B_2, \ldots X_n \in B_n\})\end{aligned}$$

Theorem 4.8.3 (continued 6)

Proof (continued). ...

$$\begin{aligned} &= \big(P(\{X_1 \in (a_1^1, b_1^1]\}) + P(\{X_1 \in (a_1^2, b_1^2]\}) + \cdots \\ &\quad + P(\{X_1 \in (a_1^\ell, b_1^\ell]\})\big) P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\}) \\ &= P(\{X_1 \in B_1\}) P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\}), \end{aligned}$$

so $\mathcal{C}$ includes the field of all finite unions of right-semiclosed intervals. So, by the Monotone Class Theorem, $\mathcal{C}$ includes the σ-field generated by the field of all finite unions of right-semiclosed intervals (which, by the Royden and Fitzpatrick Exercise 1.36 is the Borel sets in $\mathbb{R}$). So (1) holds if B_1 is any set.

Theorem 4.8.3 (continued 6)

Proof (continued). ...

$$\begin{aligned} &= \big(P(\{X_1 \in (a_1^1, b_1^1]\}) + P(\{X_1 \in (a_1^2, b_1^2]\}) + \cdots \\ &\quad + P(\{X_1 \in (a_1^\ell, b_1^\ell]\})\big) P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\}) \\ &= P(\{X_1 \in B_1\}) P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\}), \end{aligned}$$

so $\mathcal{C}$ includes the field of all finite unions of right-semiclosed intervals. So, by the Monotone Class Theorem, $\mathcal{C}$ includes the σ-field generated by the field of all finite unions of right-semiclosed intervals (which, by the Royden and Fitzpatrick Exercise 1.36 is the Borel sets in $\mathbb{R}$). So (1) holds if B_1 is any set. We can now inductively show that (1) holds for any $B_2, B_3, \ldots, B_n$ Borel sets. (As the text says, "Explicitly, we prove by induction that if $B_1, \ldots, B_i$ are arbitrary Borel sets and $B_{i+1}, \ldots, B_n$ are right-semiclosed intervals, then (1) holds for $B_1, \ldots, B_n$.") □

Theorem 4.8.3 (continued 6)

Proof (continued). ...

$$\begin{aligned} &= \Big(P(\{X_1 \in (a_1^1, b_1^1]\}) + P(\{X_1 \in (a_1^2, b_1^2]\}) + \cdots \\ &\quad + P(\{X_1 \in (a_1^\ell, b_1^\ell]\})\Big) P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\}) \\ &= P(\{X_1 \in B_1\}) P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\}), \end{aligned}$$

so $\mathcal{C}$ includes the field of all finite unions of right-semiclosed intervals. So, by the Monotone Class Theorem, $\mathcal{C}$ includes the σ-field generated by the field of all finite unions of right-semiclosed intervals (which, by the Royden and Fitzpatrick Exercise 1.36 is the Borel sets in $\mathbb{R}$). So (1) holds if B_1 is any set. We can now inductively show that (1) holds for any $B_2, B_3, \ldots, B_n$ Borel sets. (As the text says, "Explicitly, we prove by induction that if $B_1, \ldots, B_i$ are arbitrary Borel sets and $B_{i+1}, \ldots, B_n$ are right-semiclosed intervals, then (1) holds for $B_1, \ldots, B_n$.") □