0.3. Independence and Conditional Probabilities

Note. In this section, we define independent events and conditional probability. We state the Law of Total Probability and Bayes' Formula. It is understood that all results concern the probability space $(\Omega, \mathcal{F}, P)$.

Definition. Two events A and B are *independent* if the probability of their intersection equals the product of their individual (or "marginal") probabilities: $P(A \cap B) = P(A) \cdot P(B)$. For finite collection of events $\{A_1, A_2, \ldots, A_n\}$ with $P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$ for $i \neq j$, the collection of events are *pairwise independent*.

Definition. For two events A and B with $P(B) > 0$, the *conditional probability* of A given B, denoted $P(A \mid B)$, is $P(A \mid B) = P(A \cap B)/P(B)$.

Note. If A and B are independent, then $P(A \mid B) = (P(A) \cdot P(B))/P(B) = P(A)$ as expected. We conclude this brief section with the law of total probability and Bayes' Formula.

Theorem 0.3.A. Let $\{H_k \mid 1 \leq k \leq n\}$ be a partition of Ω (so that the H_k are pairwise disjoint sets whose union is Ω). For A any event, the Law of Total Probability states that

$$P(A) = \sum_{k=1}^{n} P(A \mid H_k) \cdot P(H_k).$$

Bayes' Formula states that

$$P(H_i \mid A) = \frac{P(A \mid H_i) \cdot P(H_i)}{\sum_{k=1}^{n} P(A \mid H_k) \cdot P(H_k)}.$$

Note. The Law of Total Probability and Bayes' Formula are discussed in Foundations of Probability and Statistics-Calculus Based (MATH 2050) in Section 2.3. Conditional Probability and Independence. Detailed proofs are given in Mathematical Statistics 1 (STAT 4047/5047) in Section 1.4. Conditional Probability and Independence (see Theorem 1.4.B and Theorem 1.4.1).

Note. Thomas Bayes was born on the outskirts of London and graduated in 1719 from the University of Edinburgh where he studied logic and theology. He became a minister in a Presbyterian chapel near London. Bayes published his theory of probability in "Essay Towards Solving a Problem in the Doctrine of Chances," *Philosophical Transactions of the Royal Society* of London in 1764. He was elected a Fellow of the Royal Society in 1742, even though he never published in math in his lifetime under his own name, although he did work in the foundations of calculus (on "the theory of fluxions" and some work on series). This biograpical information (and the following image) is from the MacTutor History of Mathematics Archive biography of Thomas Bayes.

Thomas Bayes

1702–April 17, 1761

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