0.6. Joint Distributions and Independence

Note. In this section, we consider the joint distribution function of random variables. We also define independence in this case and give some equivalent conditions to independence.

Definition. Let X and Y be random variables of the same kind (both discrete or both continuous). The *joint distribution function* is

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y) \text{ where } x, y \in \mathbb{R}.$$

In the discrete case the *joint probability function* is

$$p_{X,Y}(x,y) = P(X = x,\ Y = y) \text{ where } x, y \in \mathbb{R}.$$

In the continuous case, the *joint density* is

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \text{ where } x, y \in \mathbb{R}.$$

Definition. Let X and Y be random variables. They are *independent* if

$$P(\{X \leq x) \cap \{Y \leq y\}) = P(\{X \leq x\})P(\{Y \leq y\}) \text{ for } x, y \in \mathbb{R};$$

that is, $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ for all $x, y \in \mathbb{R}$.

Note. We have defined independence in terms of sets of the form $(-\infty, x]$. As observed in Note 0.4.A, this is sufficient for the types of sets with which we deal (namely, those in the σ-algebra generated by such intervals). Notice that independence when both X and Y are discrete is equivalent to the condition $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for all $x, y \in \mathbb{R}$, and when both X and Y are continuous it is equivalent to $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $x, y \in \mathbb{R}$. In Chapter 1, "Multivariate Random Variables," we extend these ideas from two variables (the "bivariate" case) to several variables.

Note. These ideas are addressed in Foundations of Probability and Statistics-Calculus Based (MATH 2050) in Section 2.6. Jointly Distributed Random Variables. They are addressed in Mathematical Statistics 1 (STAT 4047/5047) in Section 2.1. Distributions of Two Random Variables.

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