0.8. Limit Theorems

Note. In this section, we state the Weak Law of Large Numbers, the Central Limit Theorem, Markov's Inequality, and Chebyshev's Inequality. We present the version which is stated (and proved) in Mathematical Statistics 1 (STAT 4057/5057).

Note. The Weak Law of Large Numbers is addressed in Mathematical Statistics 1 in Section 5.1. Convergence in Probability, where Theorem 5.1.1 states:

> **Weak Law of Large Numbers.**
>
> Let $\{X_n\}$ be a sequence of independent and identically distributed ("iid") random variables having common mean $\mu < \infty$ and variance $\sigma^2 < \infty$. Let $\overline{X}_n = (\sum_{i=1}^n X_i)/n$ (this is the *sample mean*). Then the sequence $\{\overline{X}_k\}$ converges in probability to μ: $\overline{X}_n \xrightarrow{P} \mu$.

This is the same version of the Law of Large Numbers we use in these notes in see Section 6.5. The Law of Large Numbers and the Central Limit Theorem. See the Mathematical Statistics 1 notes mentioned above for other versions of the Laws of Large Numbers (such as the Kolmogorov Strong Law of Large Numbers).

Note. The Central Limit Theorem is addressed in Mathematical Statistics 1 in Section 5.3. Central Limit Theorem, where Theorem 5.3.1 states:

> **Central Limit Theorem.**
>
> Let $X_1, X_2, \ldots, X_n$ denote the observations of a random sample from a distribution that has mean μ and positive variance σ^2. Then the random variable $Y_n = (\sum_{i=1}^n -n\mu)/(\sqrt{n}\sigma) = \sqrt{n}(\overline{X}_n - \mu)/\sigma$ converges in distribution to a random variable that has a nromal distribution with mean 0 and variance 1.

The version of the Law of Large Numbers we use in these notes in see Section 6.5. The Law of Large Numbers and the Central Limit Theorem involves the condition of independence and identical in distribution ("iid") for $X_1, X_2, \ldots, X_n$. See the Mathematical Statistics 1 notes mentioned above for another version of the Central Limit Theorem (which involves iid k-dimensional random vectors).

Note. One can use the Central Limit Theorem to approximate the Binomial Distribution with an appropriate Normal Distribution. This is addressed in the Mathematical Statistics 1 notes mentioned above in Example 5.3.3. The Binomial Distribution can also be approximated with an appropriate Poisson Distribution. This is discussed in Example 5.2.6 of the book on which my online Mathematical Statistics 1 notes are based, Robert Hogg, Joseph McKean, and Allan Craig's *Introduction to Mathematical Statistics*, 8th Edition (Pearson, 2019), though Example 5.2.6 is not in my online notes.

Note. Markov's Inequality and Chebyshev's Inequality are stated and proved in Mathematical Statistics 1 (STAT 4047/5047) in Section 1.10. Important Inequalities, where they are stated as Theorems 1.10.2 and 1.10.3, respectively:

> **Markov's Inequality.**
>
> Let $u(X)$ be a nonnegative function of random variable X. If $E[u(X)]$ exists then for every positive constant c, $P(u(x) \geq c) \leq E[u(X)]/c$.

Chebyshev's Inequality.

Let X b a random variable where $E(X^2) < \infty$ (so that μ and σ^2 are defined). Then for every $k > 0$,

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2} \text{ or } P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

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