Section 4.3. Independence

Note. When we refer to events and their probability, it is understood that the discussion is in the setting of a probability space.

Definition 4.3.1. Two events A and B are *independent* if $P(A \cap B) = P(A)P(B)$.

Definition 4.3.2. Let I be an arbitrary index set, and let A_i, $i \in I$, be events. The A_i are *independent* if for all finite collections $\{i_1, i_2, \ldots, i_k\}$ of distinct indices in I, we have

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k}).$$

Lemma 4.3.A. If events A_i, $i \in I$, are independent and any event is replaced by its complement, then independence is maintained.

Note. The text gives an example of events A, B, C where the events are pairwise independent but all three events are not independent. The random experiment is to toss a coin twice. The sample space is $\Omega = \{HH, HT, TH, TT\}$ and the probability of each element of Ω is $1/4$ (technically, the probability of each one element subset of ω is $1/4$). Consider the events

$$A = \{\text{first toss is a head}\} = \{HH, HT\},$$
$$B = \{\text{second toss is a head}\} = \{HH, TH\}, \text{ and}$$
$$C = \{\text{first and second toss are some}\} = \{HH, TT\}.$$

Notice that $P(A) = P(B) = P(C) = 1/2$. Then

$$P(A \cap B) = P(\{HH\}) = \tfrac{1}{4} = \tfrac{1}{2} \times \tfrac{1}{2} = P(A)P(B),$$
$$P(A \cap C) = P(\{HH\}) = \tfrac{1}{4} = \tfrac{1}{2} \times \tfrac{1}{2} = P(A)P(C), \text{ and}$$
$$P(B \cap C) = P(\{HH\}) = \tfrac{1}{4} = \tfrac{1}{2} \times \tfrac{1}{2} = P(B)P(C).$$

So A, B, and C are pairwise independent. But

$$P(A \cap B \cap C) = P(\{HH\}) = \frac{1}{4} \neq \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = P(A)P(B)P(C),$$

So that events A, B, C are NOT independent. Conversely, Ash and Doleans-Dade give an example where events $A_1, A_2, \ldots, A_n$ are independent, but some subcollection $A_{i_1}, A_{i_2}, \ldots, A_{I_k}$ (where $k < n$) of events are NOT independent.

Note 4.3.A. Converse to the previous note, we have events A, B, C which are independent, but these events may not be independent, but these events may not be independent when taken two at a time. Consider the random experiment of rolling a six-sided die twice. Let

$$A = \{\text{first roll is 1, 2, or 5}\}$$
$$B = \{\text{first die is 4, 5, or 6}\}, \text{ and}$$
$$C = \{\text{the sum of the rolls is 9}\}.$$

Notice that $P(A) = 1/2$, $P(B) = 1/2$, $P(C) = 3/36 = 1/9$, and

$$P(A \cap B \cap C) = P(\{\text{first roll is 5 and sum is 9}\}) = 1/36$$

$$= (1/2)(1/2)(1/9) = P(A)P(B)P(C),$$

so A, B, C are independent. But

$$P(A \cap B) = P(\{\text{first roll is } 5\}) = \tfrac{1}{6} \neq \tfrac{1}{2} \times \tfrac{1}{2} = P(A)P(B),$$
$$P(A \cap C) = P(\{\text{first roll is 5 and sum is } 9\}) = \tfrac{1}{6} \neq \tfrac{1}{2} \times \tfrac{1}{9} = P(A)P(C), \text{ and}$$
$$P(B \cap C) = P(\{\text{first roll is 4, 4, 6 and sum is } 9\}) = \tfrac{3}{36} = \tfrac{1}{12} \neq \tfrac{1}{2} \times \tfrac{1}{9} = P(B)P(C).$$

So A and B are not independent, A and C are not independent, and B and C are not independent.

Revised: 2/3/2019