Section 4.4. Bernoulli Trials

Note/Definition. A sequence of n *Bernoulli trials* consists of n independent observations such that each observation has only two possible results, called "success" and "failure." The probability of a success on any trial is p and the probability of a failure is $q = 1 - p$.

Note. For an elementary discussion of Bernoulli trials, see my online Introduction to Probability and Statistics (MATH 1530) notes on "Binomial Distributions" at: `http://faculty.etsu.edu/gardnerr/1530/Chapter13.pdf`.

Note. We need to create a probability space where the sample space consists of all possible outcomes of performing n Bernoulli trials.

Note. Let Ω be all 2^n binary n-tuples. We interpret a 1 in position k as a success in the kth trial, and a 0 in position k as a failure in the kth trial. For $\omega \in \Omega$ where ω has a 1 in positions $i_1, i_2, \ldots, i_k$ and 0 in positions $i_{k+1}, i_{k+2}, \ldots, i_n$ (so ω contains k 1's and $n - k$ 0's). Let A_i be the event of obtaining a success on trial i:

$$A_i = \{\omega \in \Omega \mid \omega \text{ has a 1 in the } i\text{th position}\}.$$

Then

$$\{\omega\} = A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} \cap A^c_{i_{k+1}} \cap A^c_{i_{k+2}} \cap \cdots \cap A^c_{i_n}.$$

Since the trials are independent, $P(A_i) = p$ and $p(A^c_i) = 1 - p = q$ for all i, then

$$P(\{\omega\}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})P(A^c_{i_{k+1}})P(A^c_{i_{k+2}}) \cdots P(A^c_{i_n}) = p^k q^{n-k}.$$

We now confirm that $P(\Omega) = 1$. By elementary combinatorics, there are $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ having k 1's and $n-k$ 0's. So the probability of a randomly chosen sequence representing exactly k successes is (by the addition rule for disjoint events),

$$p(k) = \binom{n}{k} p^k q^{n-k} = \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

where $k = 0, 1, \ldots, n$. By the Binomial Theorem,

$$\sum_{\omega \in \Omega} P(\{\omega\}) = \sum_{k=0}^{n} p(k) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p_q)^n = 1^n = 1.$$

So $(\Omega, \mathcal{P}(\Omega), P)$ is a measure space where $P(\{\omega\})$ is as defined above (and P is extended to $\mathcal{P}(\Omega)$ using the addition rule for disjoint events).

Note/Definition. A sequence of n *generalized Bernoulli trials* consists of n independent observations such that each observation has k possible results (where $k > 2$) labeled $b_1, b_2, \ldots, b_k$ where result b_i occurs with probability p_i, and $\sum_{i=1}^{k} p_i = 1$.

Note. Now we create a measure space for the generalized Bernoulli trials. Let Ω be the set of all k^n ordered n-tuples with entries $b_1, b_2, \ldots, b_k$. Similar to the binary case, for $\omega \in \Omega$ where ω has n_i occurrences of b_i (for $i = 1, 2, \ldots, k$) is $P(\{\omega\}) = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$. Then using the addition rule for disjoint events we extend P to $P(\Omega)$. We now confirm that $P(\Omega) = 1$. The number of $\omega \in \Omega$ in which b_i occurs exactly n_i times can be calculated as follows. There are $\binom{n}{n_1}$ possible positions for b_1, then $\binom{n-n_1}{n_2}$ possible positions for b_2, and in general $\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$ possible positions for b_k where $1 \le k \le n$. Multiplying, the total number of such

ω is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-2}}{n_{k-1}}\binom{n_k}{n_k}$$
$$= \frac{n!}{(n-n_1)!n_1!}\frac{(n-n_1)!}{(n-n_1-n)2)!n_2!}\frac{(n-n_1-n_2)!}{(n-n_1-n_2-n_3)!n_3!}\cdots$$
$$\cdots\frac{(n-n_1-n_2-\cdots-n_{k-2})!}{n_k!n_{k-1}!}\frac{n_k!}{0!n_k!} = \frac{n!}{n_1!n_2!\cdots n_k!}$$

(cancelling $(n-n_1-n_2-\cdots-n_i)!$ in the denominator of the ith term with the $(n-n_1-n_2-\cdots-n_i)!$ in the numerator of the next term for $i=1,2,\ldots,k$). So with $A=\{\omega\in\Omega \mid \omega \text{ has } n_i \text{ occerrences of } b_i\}$, then

$$P(A) = \frac{n!}{n_1!n_2!\cdots n_k!}p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}.$$

Then by the Multinomial Theorem,

$$P(\Omega) = \sum\frac{n!}{n_1!n_2!\cdots n_k!}p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k} = (p_1+p_2+\cdots+p_k)^n = 1^n = 1$$

where the sum is taken over all n_i with $0\le n_i\le k$ and $n_1+n_2+\cdots+n_k=n$. So $(\Omega,\mathcal{P}(\Omega),P)$ is a measure space where P is defined above.

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