Section 4.5. Conditional Probability

Note. Recall that events A and B are independent if $P(A \cap B) = P(A)P(B)$. If $P(A \cap B) \neq P(A)P(B)$ then, in practice the occurrence of event A modifies the probability of the occurrence of event B (or conversely). That is, the condition of one event having occurred affects the probability of the occurrence of a future dependent event.

Definition. Let A and B be events in a measure space. The *conditional probability of B given A* is

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \text{ provided } P(A) > 0.$$

Note. In Chapter 5 we address the case of a conditional probability given event A where $P(A) = 0$. This section only considers the case where $P(A) > 0$.

Theorem 4.5.1.

(a) Let $P(A) > 0$. Events A and B are independent if and only if $P(B \mid A) = P(B)$.

(b) Let $P(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) > 0$. Then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

Theorem 4.5.2. Theorem of Total Probability.

Let $B_1, B_2, \ldots$ form a finite or countably infinite family of mutually exclusive and exhaustive events; that is, $\uplus_i B_i = \Omega$.

(a) If A if any event, then $P(A) = \sum_i P(A \cap B_i)$. Thus $P(A)$ is calculated by making a list of mutually exclusive exhaustive ways in which A can happen, and adding the individual probabilities.

(b) $P(A) = \sum_i P(B_i)P(A \mid B_i)$ where the sum is taken over those i for which $P(B_i) > 0$. Thus $P(A)$ is a weighted average of the conditional probabilities $P(A \mid B_i)$.

Note. The book gives an example involving conditional probability which requires summing a series. Instead, we present an example related to false positives in medical tests. This example is based on a Brown.com website (see `https://brownmath.com/stat/falsepos.htm`), which is itself based on John Paulos' *A Mathematician Reads the Newspaper* (pages 136–137). They state:

> Suppose you're told the test for D is "99% accurate" in the following sense: If you have D, the test will be positive 99% of the time, and if you dont have it, the test will be negative 99% of the time. The other 1% in each scenario would be a false negative or false positive. (For simplicity, Im using the same percentage for both positive and negative results. Many tests have a different accuracy for positive and negative.) Suppose further that 0.1% — one out of every thousand people — have this rare disease. You might think that a positive result means you're 99% likely to have the disease. But 99% is the probability that if you have the disease then you test positive, not the probability that if you test positive then you have the disease. ...Suppose 100,000 [random] people are tested for disease D.

We want to calculate the probability that a random person in the 100,000 people has disease D given they test positive for it. Let P be the set of individuals that test positive, D the set of individuals with the disease (we are given above that $P(D) = 0.001$), H be the set of individuals that don't have the disease (so $P(H) = 1 - P(D) = .999$), P the set of individuals that test positive, and N the set of individuals that test negative. Since the test is 99% "accurate," we have

$$P(\text{test positive} \mid \text{have the disease}) = P(P \mid D) = 0.99$$

and

$$P(\text{test positive} \mid \text{don't have the disease}) = P(P \mid H) = 0.01.$$

Now

$$P(\text{have the disease} \mid \text{test positive}) = P(D \mid P) = \frac{P(D \cap P)}{P(P)} = \frac{P(D)P(P \mid D)}{P(P)}.$$

We need $P(\text{test positive}) = P(P)$. By Theorem 4.5.2(b),

$$P(P) = P(D)P(P \mid D) + P(H)P(P \mid H) = (0.001)(0.99) + (0.999)(0.01) \approx .011.$$

Then

$$P(D \mid P) = \frac{P(D)P(P \mid D)}{P(P)} \approx \frac{(0.001)(0.99)}{(0.011)} = 0.09.$$

So, *in this population*, the probability that a random individual has disease D given that they test positive for it is about 9%. In addition, the probability that an individual who tests positive has the disease is 90% (this is the probability of a false positive). Of course, people are not usually tested at random for a disease, but are more likely to have symptoms and get tested for a specific disease based on symptoms. However in the large-scale testing of a population with no prior evidence of "symptoms" (such as in large scale drug testing), the concern of a false positive result is legitimate.